Fluctuation theory of Markov additive processes and self-similar Markov processes

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Abstract

By a self-similar process we mean a stochastic process having the scaling property. Self-similar processes often arise in various areas of probability theory as limit of re-scaled processes. Among several classes of self-similar processes, of particular interest to us is the class of self-similar strong Markov processes (ssMp).

The ssMp's are involved for instance in branching processes, Lévy processes, coalescent processes and fragmentation theory. Some particularly well-known examples are Brownian motion, Bessel processes, stable subordinators, stable processes, stable Lévy processes conditioned to stay positive, etc. Our main purpose in this course is to give a panorama of properties of ssMp's that have been obtained since the early sixties under the impulse of Lamperti's work, where the study of the case of positive valued ssMp's is initiated. The main result in Lamperti's work establishes that there is an explicit bijection between positive valued ssMp's and real valued Lévy processes. Recently it has been proved by Alili et al. that $\mathbb{R}^d$-valued ssMp's are in a bijection with a generalization of Lévy processes, namely *Markov Additive Processes* (MAP).

In this course we will mainly focus in the study of ssMp's making a systematic application of the fluctuation theory of Lévy processes and MAP's. So, we will start by giving a review of some key results in the fluctuation theory of Lévy processes and random walks, and then extending some of those results to MAP's. We will study some particular examples, most of them are ssMp's obtained as a path transformation of stable processes.

1 Motivation: Scaling limits during 50's-60's

Donsker theorem. One of the first results one learns from the theory of continuous time stochastic processes is that Brownian motion can be approximated by a random walk, both in the sense of finite dimensional distributions and in that of Skohorod topology. In dimension one, the approximation works as follows. Let $\{Y_i, i \geq 1\}$ be i.i.d. real valued random variables such that

$$\mathbb{E}(Y_i) = 0, \qquad \mathrm{Var}(Y_i) = \sigma^2 > 0. \tag{1.1}$$

We denote

$$S_m = \sum_{k=1}^{m} Y_k, \qquad m \geq 0, \tag{1.2}$$

$$\widetilde{X}_t^{(n)} = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) Y_{\lfloor t \rfloor + 1}, \qquad 0 \leq t < \infty \tag{1.3}$$

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} \widetilde{X}_{nt}^{(n)} = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + \frac{nt - \lfloor nt \rfloor}{\sigma\sqrt{n}} Y_{\lfloor nt \rfloor + 1}, \qquad 0 \leq t < \infty. \tag{1.4}$$

It is an elementary consequence of the central limit theorem and the simple Markov property of random walks that

$$(X_t^{(n)}, 0 \leq t < \infty) \underset{n\to\infty}{\to} (B_t, 0 \leq t < \infty) \tag{1.5}$$

in the senses both of convergence of finite-dimensional distributions (which we will call the *fdd* convergence. It is slightly more complicated to establish the weak convergence in the Skorohod topology, but it is part of any first course on weak convergence. The most important fact to deduce from this approximation is that Brownian motion has the scaling property: for any $c > 0$ and $x \in \mathbb{R}$,

$$(cB_{c^{-2}t}, t \geq 0) \text{ with } B_0 = x \quad \overset{\mathcal{L}}{=} \quad (B_t, t \geq 0) \text{ with } B_0 = cx. \tag{1.6}$$

The time interval can be extended to $0 \leq t < \infty$.

Exercise 1.1. Check that the scaling property from the limit.

Extremal process. Let $(Y_i, i \geq 1)$ be i.i.d. random variables such that $Y_i \geq 0$ and the distribution of Y_i is heavy tailed, i.e., there exists $\alpha > 0$ and a positive measurable function l such that

$$\overline{F}(x) = 1 - F(x) = \mathbb{P}(Y_1 > x) = x^{-\alpha} l(x), \tag{1.7}$$

$$\frac{l(cx)}{l(x)} \underset{x\to\infty}{\to} 1, \text{ for all } c > 0, \tag{1.8}$$

said otherwise, ℓ, is slowly varying. We denote

$$M_k = \max\{Y_i, i \leq k\}, \tag{1.9}$$

$$X_n(t) = \begin{cases} \frac{M_{\lfloor nt \rfloor}}{a_n}, & t \geq \frac{1}{n}, \\ \frac{M_1}{a_n}, & t < \frac{1}{n}, \end{cases} \tag{1.10}$$

$$a_n = \inf\left\{ s > 0, \overline{F}(s) < \frac{1}{n} \right\}, \qquad \text{for all } n > 0. \tag{1.11}$$

$$\tag{1.12}$$

We then have that $\overline{F}(a_n) \sim \frac{1}{n}$, as $n \to \infty$.

Lamperti(1962)[29] and Durrett–Resnick(1978)[19] proved that

$$(X_n(t), t \geq 0) \underset{n\to\infty}{\to} (X_t, t \geq 0), \tag{1.13}$$

in the sense of fdd convergence and of weak convergence. The limit X is a process whose finite-dimensional distributions are given by

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_k} \leq x_k) = \varphi^{t_1}(\widehat{x}_1)\varphi^{t_2-t_1}(\widehat{x}_2)\cdots\varphi^{t_k-t_{k-1}}(\widehat{x}_k), \tag{1.14}$$

for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < \infty$ and $x_1, x_2, \cdots, x_k \in (0, \infty)$ where

$$\varphi^t(x) = \exp\{-tx^{-\alpha}\}, \quad t, x > 0, \qquad \widehat{x}_n = \bigwedge_{i=n}^{k} x_i. \tag{1.15}$$

Exercise 1.2. Prove that the process X, whose finite dimensional distributions are given by (1.15) has the scaling property: for any $c > 0$,

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ with } X_0 = x \quad \overset{\mathcal{L}}{=} \quad (X_t, t \geq 0) \text{ with } X_0 = cx. \tag{1.16}$$

We check (1.15) for $k = 1$. For $t > 0$, $x \geq 0$ and large enough $n > 0$,

$$\begin{aligned}
\mathbb{P}(X_n(t) \leq x) &= \mathbb{P}(M_{\lfloor nt \rfloor} \leq xa_n) && (1.17)\\
&= \mathbb{P}(Y_1 \leq xa_n, \dots, Y_{\lfloor nt \rfloor} \leq xa_n) && (1.18)\\
&= \mathbb{P}(Y_1 \leq xa_n)^{\lfloor nt \rfloor} && (1.19)\\
&= (1 - \overline{F}(xa_n))^{\lfloor nt \rfloor} && (1.20)\\
&= (1 - (xa_n)^{-\alpha} l(xa_n))^{n\frac{\lfloor nt \rfloor}{n}} && (1.21)\\
&= \left(1 - x^{-\alpha}{a_n}^{-\alpha} l(a_n)\frac{l(xa_n)}{l(a_n)}\right)^{n\frac{\lfloor nt \rfloor}{n}}. && (1.22)
\end{aligned}$$

Since we know that

$$\frac{\lfloor nt \rfloor}{n} \sim t, \qquad {a_n}^{-\alpha} l(a_n) = \overline{F}(a_n) \sim \frac{1}{n} \qquad \text{and} \qquad \frac{l(xa_n)}{l(a_n)} \sim 1, \tag{1.23}$$

we have

$$\mathbb{P}(X_n(t) \leq x) \sim \left(1 - \frac{x^{-\alpha}}{n}\right)^{nt} \sim e^{-x^{-\alpha}t}. \tag{1.24}$$

For $k = 2$ with $x_1 \leq x_2$, we have

$$\begin{aligned}
&\mathbb{P}(X_n(t_1) \leq x_1, X_n(t_2) \leq x_2) = \mathbb{P}(X_n(t_1) \leq x_1)\, \mathbb{P}(X_n(t_2) \leq x_2 | X_n(t_1) \leq x_1) && (1.25)\\
&= \mathbb{P}(X_n(t_1) \leq x_1)\, \frac{\mathbb{P}(Y_j \leq x_2 a_n, \lfloor nt_1 \rfloor < j \leq \lfloor nt_2 \rfloor; Y_i \leq x_1 a_n, 1 \leq i \leq \lfloor nt_1 \rfloor)}{\mathbb{P}(Y_i \leq x_1 a_n, 1 \leq i \leq \lfloor nt_1 \rfloor)} && (1.26)\\
&= \mathbb{P}(X_n(t_1) \leq x_1)\, \mathbb{P}(Y_j \leq x_2 a_n, \lfloor nt_1 \rfloor < j \leq \lfloor nt_2 \rfloor) && (1.27)\\
&= \mathbb{P}(X_n(t_1) \leq x_1)\, \mathbb{P}(Y_i \leq x_2 a_n, i \in (0, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor]) && (1.28)\\
&\to \varphi^{t_1}(x_1)\varphi^{t_2-t_1}(x_2), && (1.29)
\end{aligned}$$

as $n \uparrow \infty$. The convergence of the finite dimensional distributions is established recursively using the above argument.

In 60's many other examples of scaling limits appeared in other areas of stochastic modeling, as for instance Renewal theory, branching properties and others. All share the scaling property. Lamperti noticed the common fact that the resulting limit process bears the scaling property. This allowed Lamperti to establish that the class of self-similar processes is exactly that of scaling limits of stochastic processes. To provide a precise statement we first provide a definition.

Definition 1.3. Let $(X_t, t \geq 0) =: X$ be defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), (\mathbb{P}_x, x \in \mathbb{R}^d))$ taking values in $\mathbb{R}^d$. We will say that X is *self-similar* if there exists $\alpha \in \mathbb{R}$ such that for any $c > 0$ and $x \in \mathbb{R}^d$,

$$(\{cX_{tc^{-\alpha}}, t \geq 0\}, \mathbb{P}_x) \stackrel{\mathcal{L}}{=} (\{X_t, t \geq 0\}, \mathbb{P}_{cx}). \tag{1.30}$$

If $\alpha \neq 0$, we will say that X is $\frac{1}{\alpha}$-self-similar.

Example 1.4. Brownian motion is $\frac{1}{2}$-self-similar. The extremal process is $\frac{1}{\alpha}$-self-similar. Bessel process is $\frac{1}{2}$-self-similar. α-stable processes are $\frac{1}{\alpha}$-self-similar. All of these processes are Markov processes. Fractional Brownian motion is also self-similar but may not be Markov.

Theorem 1.5 (Lamperti(1962)[29])**.** *Let $(Y_t, t \geq 0)$ be a process in $\mathbb{R}^d$, $f : [0, \infty) \to \mathbb{R}$ be a measurable map with $f(\xi) \neq 0$ for all ξ, and define Y^ξ by*

$$Y_t^\xi = \frac{Y_{t\xi}}{f(\xi)}, \quad t \geq 0. \tag{1.31}$$

If Y^ξ has a non-degenerate process, X, as a limit as $\xi \to \infty$, in the sense of the fdd convergence, i.e.

$$(Y_{t_1}^\xi, \dots, Y_{t_n}^\xi) \underset{\xi \to \infty}{\stackrel{\mathcal{L}}{\to}} (X_{t_1}, \dots, X_{t_n}), \tag{1.32}$$

for any $0 < t_1 < t_2 < \cdots < t_n < \infty$, and $n \geq 1$, then, for some $\alpha \in \mathbb{R}$, X is a self-similar process with index $\alpha \in \mathbb{R}$, and f is a regularly varying function at infinity with index α. (i.e. $f(\xi) = \xi^\alpha l(\xi)$ for some slowly varying function l.)

The proof uses Lamperti's theorem of types as follows.

Theorem 1.6. *Assume $(\varphi_n, n \geq 1)$ is a sequence of distribution functions, F_1, F_2 are non-degenerate distributions and $(a_n, n \geq 0)$, $(b_n, n \geq 0)$ are sequences of positive numbers such that*

$$\varphi_n(a_n x) \underset{n \uparrow \infty}{\to} F_1(x), \tag{1.33}$$

$$\varphi_n(b_n x) \underset{n \uparrow \infty}{\to} F_2(x), \tag{1.34}$$

for continuity points x of F_1 and F_2. Then we have

$$0 < \lim_{n \uparrow \infty} \frac{a_n}{b_n} < \infty. \tag{1.35}$$

Using this and results from the theory of regularly varying functions (see e.g. [10]), one can easily verify that X should have the scaling property, and that f is regularly varying. For it is enough to verify that for any $c > 0$,

$$\lim_{\xi \uparrow \infty} \frac{f(c\xi)}{f(\xi)} \in (0, \infty). \tag{1.36}$$

As a refinement of Lamperti's characterisation of the class of self-similar processes as scaling limits, Haas–Miermont(2011)[22] and Bertoin–Kortchemski(2016)[6] studied scaling limits of Markov chains. Indeed, the provided sufficient conditions for it to have a scaling limit that is a non-degenerated pssMp. We next quote the main result in the former paper, which deals with non-increasing processes.

Let $(X(k), k \geq 0)$ be a discrete-time Markov chain taking values in $\{0, 1, 2, \dots\}$ with transition probabilities

$$\mathbb{P}(X(l+1) = k | X(l) = n) = p_{n,k}, \qquad 0 \leq k \leq n, \tag{1.37}$$

with

$$\sum_{k=0}^{n} p_{n,k} = 1 \tag{1.38}$$

for all $n \geq 0$. We will denote by $(X_n(l), l \geq 0)$ the Markov chain issued from $X_n(0) = n$ and by p_n^* the law on $[0, 1]$ of $X_n(1)$:

$$p_n^*(dx) = \sum_{0 \leq k \leq n} p_{n,k} \delta_{\frac{k}{n}}(dx), \quad x \in [0, 1]. \tag{1.39}$$

Theorem 1.7 (Haas–Miermont(2011)[22])**.** *Assume there is a sequence*

$$a_n = n^{\gamma} l(n), \quad n \geq 0, \tag{1.40}$$

for some $\gamma > 0$, a slowly varying function l and a finite measure μ on $[0, 1]$ such that

$$a_n(1 - x) p_n^*(dx) \xrightarrow[n \to \infty]{\mathrm{w}} \mu(dx). \tag{1.41}$$

Then the sequence of processes $(Y_n, n \geq 0)$ defined as

$$Y_n(t) = \frac{X_n([a_n t])}{n}, \quad t \geq 0, n \geq 0 \tag{1.42}$$

converges in the sense of weak convergence in the Skorohod topology towards a self-similar Markov process with non-increasing paths, say Z. Furthermore, Z can be represented as

$$Z_s = \exp(-\sigma_{\tau(s)}), \qquad s \geq 0, \tag{1.43}$$

where σ is a subordinator (a Lévy process with non-decreasing paths) and

$$\tau(s) = \inf \left\{ u > 0 : \int_0^u e^{-\gamma \sigma_v} dv > s \right\}. \tag{1.44}$$

σ has a Laplace exponent ϕ, related to μ, via the formula

$$\phi(\lambda) = \int_{[0,1]} \frac{1 - x^{\lambda}}{1 - x} \mu(dx), \qquad \lambda \geq 0.$$

They also proved that the absorption time of X_n at 0 (or at any other point), which we will denote by A_n, satisfies

$$a_n A_n \xrightarrow[n\uparrow\infty]{\mathcal{L}} \int_0^\infty e^{-\gamma\sigma_s} ds. \tag{1.45}$$

The limit is the lifetime of Z. We also have the convergence of moments:

$$\mathbb{E}\Big((a_n A_n)^k\Big) \underset{n\uparrow\infty}{\to} \mathbb{E}\left(\left(\int_0^\infty e^{-\gamma\sigma_s} ds\right)^k\right), \qquad k \geq 0. \tag{1.46}$$

2 Second Lamperti transform

In this section we would like to explain the bijection between pssMp and killed Lévy processes, since this bijection is explicitly given by the exponential of a Lévy process time changed, we will refer to it as Lamperti's transform. In some of the literature it is also referred as second Lamperti's transform, the reason for this is that there are two other transformations due to him that we now enlist.

First Lamperti transform If X is an α-self-similar process, not necessarily Markovian, started from 0, then the process $Z_t = e^{-\alpha t} X_{e^t}$, $t \in (-\infty, \infty)$ is a stationary process, and any stationary process can be obtained in this way. This transformation proved very useful in the theory of stationary processes.

Second Lamperti transform Is the main motivation of the course. It establishes that there is a bijection between positive self-similar Markov processes and Lévy processes. See the next Theorem.

Third Lamperti transform There is a bijection between continuous state branching process and spectrally positive Lévy processes. In branching processes theory this transform has allowed to establish many interesting results. See e.g. [31].

In what follows we will use the term of killed Lévy process to refer to a Lévy process killed at an independent exponential time of parameter $q \geq 0$; where we allow $q = 0$ to include the case where the life time is infinite, which coincides with the usual meaning of an exponential r.v. with parameter zero, whose unique value is infinity a.s.

Let us now recall two facts from the theory of Markov processes. We say that a process $(X_t, t \geq 0)$ $\mathbb{R}^d$-valued defined on $(\omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, (\mathbb{P}_x, x \in \mathbb{R}^d))$ has the strong Markov property if for every stopping time T of $\{\mathcal{F}_t, t \geq 0\}$ i.e. $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$, we have that the law of $\{X_{T+t}, t \geq 0\}$ conditionally on $\mathcal{F}_T \cup \{T < \infty\}$ has the same law as $\{X_t, t \geq 0\}$ starting from X_T where $\mathcal{F}_T = \{A \in \mathcal{F}, A \cup \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$. It is enough to verify that for all $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ which is measurable and bounded, we have

$$\mathbb{E}_x(f(X_{T+s})|\mathcal{F}_T) = \mathbb{E}_z(f(X_s))\,|_{z=X_T}, \qquad s \geq 0. \tag{2.1}$$

Note that X is a time-homogeneous Markov process. We assume also that X has right continuous and left limited paths (càdlàg) and that X is quasi-left-continuous i.e. if stopping times $\{T_n\}_{n\in\mathbb{N}}$ satisfy $T_n \underset{n\to\infty}{\uparrow} T$, then $X_{T_n} \underset{n\to\infty}{\to} X_T$ on $\{T < \infty\}$.

For Lévy processes, the strong Markov property can be conveniently written a follows. Let ξ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ and T is a stopping time then $(\xi_{T+s} - \xi_T, s \geq 0)$ on $\{T < \infty\}$ is independent of $(\xi_u, u \leq T)$, and has the same law as $(\xi_s, s \geq 0)$. Precisely, for any measurable and non-negative functionals F and φ,

$$\mathbb{E}\big(F(\xi_{T+s} - \xi_T, s \geq 0)1_{\{T<\infty\}}\varphi(\xi_u, u \leq T)\big) = \mathbb{E}(F(\xi_s, s \geq 0))\,\mathbb{E}\big(\varphi(\xi_u, u \leq T)1_{\{T<\infty\}}\big)\,. \tag{2.2}$$

Theorem 2.1 (Lamperti(1972)[30]). *Let $\alpha > 0$ and let $\{\xi_t, t \geq 0\}$ be a possibly killed Lévy process taking values in $\mathbb{R}$. Then we can construct a positive $\frac{1}{\alpha}$-self-similar Markov process $X^{(\alpha)}$ by defining*

$$X_t^{(x)} = x\exp\{\xi_{\tau(tx^{-\alpha})}\}1_{\{t<x^\alpha A_\infty\}}, \tag{2.3}$$

where $A_\infty = \int_0^\infty e^{\alpha\xi_s}ds$ and

$$\tau(t) = \inf\left\{s > 0 : \int_0^s e^{\alpha\xi_u}du > t\right\} \quad (t < A_\infty). \tag{2.4}$$

The cemetery state 0 for $X^{(x)}$ is identified with $-\infty$ for ξ. Moreover, we have the following three possibilities:

(C1) ξ has an infinite lifetime ($q = 0$) and $\overline{\lim}_{t\uparrow\infty}\xi_t = \infty$ a.s.
if $\mathbb{P}_x(T_0 < \infty) = 0$ for all $x > 0$.

(C2) ξ has an infinite lifetime ($q = 0$) and $\lim_{t\uparrow\infty}\xi_t = -\infty$ a.s.
if $\mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0) = 1$ for all $x > 0$.

(C3) The lifetime of ξ is finite a.s. ($q > 0$)
if $\mathbb{P}_x(T_0 < \infty, X_{T_0-} > 0) = 1$ for all $x > 0$.

There are no other possibilities. If the conditions are satisfied for some $x > 0$, then they are satisfied for all $x > 0$.

Example 2.2 (Brownian motion killed at 0 and squared Bessel processes). Let $(B_t, t \geq 0)$ be a Brownian motion and let $\xi_t = 2B_t + 2ct$, $t \geq 0$. Let X be the 1-self-similar Markov process obtained via Lamperti's transform

$$X_t^{(x)} = x\exp\{\xi_{\tau(t/x)}\}1_{\{t<xA_\infty\}}, \quad x > 0,\ t \geq 0, \tag{2.5}$$

where $\tau(s) = \inf\left\{t > 0 : \int_0^t e^{\xi_u}du > s\right\}$ for $s < A_\infty = \int_0^\infty e^{\xi_u}du$. Then $X^{(x)}$ has continuous paths and is the stopped process upon hitting 0 of a solution Z to the SDE

$$dZ_t = 2\sqrt{Z_t}d\beta_t + \delta dt, \qquad t \geq 0, \tag{2.6}$$

where β is another Brownian motion and $\delta = 2(c + 1)$, and thus Z is a squared Bessel process of dimension δ.

Exercise 2.3. Prove that the process defined by (2.5) satisfies the SDE in (2.6) by achieving the following steps:

(i) For $c \in \mathbb{R}$, apply Itô's formula for

$$Y_t = xe^{2(B_t+ct)}. \tag{2.7}$$

(ii) Use Dubins–Schwartz theorem for the martingale $M_t = \int_0^t \sqrt{x} e^{B_s+cs} dB_s$ whose quadratic variation is

$$\langle M \rangle_t = x \int_0^t e^{2(B_s+cs)} ds, \tag{2.8}$$

in order to obtain the SDE (2.6).

Is there strong uniqueness of the solution to (2.6)? Moreover, the exponential functional $\int_0^\infty e^{2(B_s+cs)} ds$ is finite a.s. iff $c < 0$ ($\delta < 2$). If $c \geq 0$ ($\delta \geq 2$), $X^{(x)}$ will not hit 0 a.s., which coincides with the well-known fact that a squared Bessel process does not hit 0 if of dimension $\delta \geq 0$.

The converse to Theorem 2.1 can be stated as follows.

Theorem 2.4. *Let* $\left\{ (X^{(x)}, \mathbb{P}) \stackrel{\mathcal{L}}{=} (X, \mathbb{P}_x), x > 0 \right\}$ *be a positive self-similar Markov process (pssMp) that dies at* 0 *at time* T_0. *Define* $C_t = \int_0^t X_s^{-\alpha} ds$ *and* $B_t = C_t^{-1} = \inf\{u \geq 0 : C_u > t\}$ *for* $t \geq 0$. *Under* $\mathbb{P}_x$, *the process*

$$\xi_t = \log \frac{X_{B_t}}{X_0}, \quad 0 \leq t < C_{T_0} \tag{2.9}$$

is a Lévy process that starts from 0, *whose law does not depend on the starting point of* X, *and the description (C1)–(C3) holds. The lifetime of* ξ *is* $C_{T_0} = \int_0^{T_0} X_s^{-\alpha} ds \sim \exp(q)$.

To get acquainted with the time change in Lamperti's transformation one can check that the law of the process $\xi_t^{(cx)}$, $t \geq 0$ does not depend on the starting point of $X_0 = x > 0$. For that end recall that by the scaling property, for any $c > 0$, the process $\widetilde{X}_t^{(cx)} := cX_{tc^{-\alpha}}^{(x)}$, $t \geq 0$, is a pssMp with the same law as X issued from cx. We denote by $\xi^{(cx)}$ the stochastic process defined by (2.9) using the process $\widetilde{X}^{(cx)}$, and in general we will put a superindex $^{(cx)}$ to any object defined using this process. We have the following

$$\begin{aligned}
B_t^{(cx)} &= \inf\left\{ s > 0 : \int_0^s \left(\widetilde{X}_u^{(cx)}\right)^{-\alpha} du > t \right\} && (2.10)\\
&\stackrel{\mathcal{L}}{=} \inf\left\{ s > 0 : \int_0^s \left(cX_{uc^{-\alpha}}^{(x)}\right)^{-\alpha} du > t \right\} && (2.11)\\
&= \inf\left\{ s > 0 : \int_0^{sc^{-\alpha}} \left(X_v^{(x)}\right)^{-\alpha} dv > t \right\} && (2.12)\\
&= c^\alpha B_t^{(x)}, && (2.13)
\end{aligned}$$

where to pass from the second to the third line we make the change of variables $v = c^{-\alpha}u$. Thus we obtain the equality in law between the processes

$$\xi_t^{(cx)} = \log \frac{X^{(cx)}_{B_t^{(cx)}}}{X_0^{(cx)}} \stackrel{\mathcal{L}}{=} \log \frac{cX^{(x)}_{B_t^{(x)}}}{cX_0^{(x)}} = \log \frac{X^{(x)}_{B_t^{(x)}}}{X_0^{(x)}} = \xi_t^{(x)}, \qquad t \geq 0. \tag{2.14}$$

3 Proof of Theorem 2.1

The first step to establish Theorem 2.1 is to verify that $X^{(x)}$ as defined in (2.3) has the scaling property. This follows straightforwardly from the remark that, for any $c > 0$,

$$\left(cX^{(x)}_{tc^{-\alpha}}, t \leq c^\alpha T_0^{(x)}\right) = \left(cxe^{\xi_{\tau(t(xc)^{-\alpha})}}, t \leq (cx)^\alpha \int_0^\infty e^{\alpha\xi_s} ds\right) = \left(X_t^{(cx)}, t \leq T_0^{(cx)}\right). \tag{3.1}$$

The strong Markov property of the process defined in (2.3) is a consequence of a result due to Volkonskiĭ(1958)[40]:

Theorem 3.1. *Let Y be a strong Markov process. Let $A_t = \int_0^t \nu(Y_u)du$, $t \geq 0$ with ν a measurable, non-negative function such that the integral is finite for all $t \geq 0$. We define*

$$c(t) = \inf\{s > 0 : A_s > t\}, \qquad t \geq 0. \tag{3.2}$$

Then the process $\widetilde{Y}$ defined by time change

$$\widetilde{Y}_t = \begin{cases} Y_{c(t)}, & \text{if } c(t) < \infty, \\ \Delta, & \text{if } c(t) = \infty, \end{cases} \qquad t \geq 0, \tag{3.3}$$

where Δ is the cemetery state for $\widetilde{Y}$, has the strong Markov property with respect to $\left(\widetilde{\mathcal{F}}_t = \mathcal{F}_{c(t)}, t \geq 0\right)$.

3.1 The trichotomy (C1)–(C3)

Let us verify that there are no other possibility than (C1)–(C3), as is stated in Theorem 2.1.

Proposition 3.2. *Let X be a pssMp. Then the following three claims hold:*

(i) Either $\mathbb{P}_x(T_0 < \infty)$ equals 1 for all $x > 0$ or it equals 0 for all $x > 0$.

(ii) Either $\mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0)$ equals 1 for all $x > 0$ or it equals 0 for all $x > 0$.

(iii) Either $\mathbb{P}_x(T_0 < \infty, X_{T_0-} > 0)$ equals 1 for all $x > 0$ or it equals 0 for all $x > 0$.

Proof. Let us prove (i). Observe that for any $c > 0$,

$$c^\alpha T_0^{(x/c)} = \inf\{t \geq 0 : X_{tc^{-\alpha}}^{(x/c)} = 0\} \stackrel{\mathcal{L}}{=} \inf\{t \geq 0 : c^{-1}X_t^{(x)} = 0\} = T_0^{(x)}. \tag{3.4}$$

So $p := \mathbb{P}_x(T_0 < \infty)$ does not depend on $x > 0$. By the simple Markov property,

$$p - \mathbb{P}_x(T_0 \leq t) = \mathbb{P}_x(t < T_0 < \infty) = \mathbb{E}_x\big(\mathbb{P}_{X_t}(T_0 < \infty)1_{\{t<T_0\}}\big) = p\mathbb{P}_x(t < T_0). \tag{3.5}$$

Letting $t \to \infty$, we obtain that $0 = p(1-p)$, which proves the claim.

We give the proof of (ii). By a similar argument as above, we have the equality in law

$$\left(c^\alpha T_0^{(x/c)}, cX_{\cdot c^{-\alpha}}^{(x/c)}\right) \stackrel{\mathcal{L}}{=} \left(T_0^{(x)}, X_\cdot^{(x)}\right). \tag{3.6}$$

So, for any $x > 0$ and $c > 0$,

$$p := \mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0) \tag{3.7}$$
$$= \mathbb{P}_{\frac{x}{c}}(c^\alpha T_0 < \infty, cX_{T_0-} = 0) \tag{3.8}$$
$$= \mathbb{P}_{\frac{x}{c}}(T_0 < \infty, X_{T_0-} = 0), \tag{3.9}$$

which shows that p does not depend on x. For $y > 0$, set $\kappa_y^- = \inf\{t > 0 : X_t < y\}$ with $\kappa_y^- = T_0$ if $\{t > 0 : X_t < y\} = \emptyset$. Then, by the strong Markov property applied at the stopping time κ_y^-, we obtain

$$p = \mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0) = \mathbb{E}_x\left(1_{\{\kappa_y^- < T_0 < \infty\}}1_{\{X_{T_0^-} = 0\}}\right) \tag{3.10}$$
$$= \mathbb{E}_x\left(1_{\{\kappa_y^- < T_0\}}\mathbb{P}_{X_{\kappa_y^-}}(T_0 < \infty, X_{T_0-} = 0)\right). \tag{3.11}$$

So $p = p\mathbb{P}_x(\kappa_y^- < T_0)$ for all $y \in (0, x)$. Suppose $p > 0$. Then $1 = \mathbb{P}_x(\kappa_y^- < T_0)$ for all $y \in (0, \infty)$. Thus X goes below any level y with probability 1, $\kappa_y^- < T_0$, $\kappa_y^- \uparrow T_0$ and by the quasi-left-continuity of X, we have that $X_{\kappa_y^-} \to X_{T_0-}$. So $p = 1$.

The proof of (iii) is similar to that of (ii), so we do not provide the details. ☐

We now prove Theorem 2.4. We should verify that for any $t, s \geq 0$ on the event $\{t + s < \zeta\}$ with ζ being the lifetime of ξ, $\xi_{t+s} - \xi_t$ is independent of $\mathcal{F}_t = \mathcal{G}_{B_t}$, where $\mathcal{G}$ is the natural filtration of X, and $B_t = \inf\{s > 0 : \int_0^s X_u^{-\alpha} du > t\}$. Before doing so, observe that for any $t > 0$ B_t is a stopping time. Indeed, we have

$$\{B_t < s\} = \left\{t < \int_0^s X_u^{-\alpha} du\right\} \in \mathcal{G}_s, \qquad s \geq 0.$$

We have

$$B_{t+s} = \inf\left\{u > 0 : \int_0^{B_t} X_u^{-\alpha} du + \int_{B_t}^u X_l^{-\alpha} dl > t + s\right\} \tag{3.12}$$
$$= B_t + \inf\left\{r > 0 : \int_0^r X_{u+B_t}^{-\alpha} du > s\right\} \tag{3.13}$$
$$= B_t + B_s \circ \theta_{B_t}, \tag{3.14}$$

where θ denotes the usual shift operator. By the strong Markov property at the stopping time B_t,

$$\text{Law of } (\{X_{B_t+u}, u \geq 0\}|\mathcal{G}_{B_t}) = \text{Law of } \{z\widetilde{X}_{uz^{-\alpha}}, u \geq 0\}|_{z=X_{B_t}}, \tag{3.15}$$

where $\widetilde{X}$ is a copy of $X^{(1)}$ independent of X_{B_t}. Take a positive measurable functional $H_t \in \mathcal{F}_t = \mathcal{G}_{B_t}$. Then

$$\begin{aligned}
\mathbb{E}\big(H_t \exp\{i\lambda(\xi_{t+s} - \xi_t)\} 1_{\{t+s<\zeta\}}\big) &= \mathbb{E}_1\left(H_t\left(\frac{X_{B_{t+s}}}{X_{B_t}}\right)^{i\lambda} 1_{\{t+s<C_{T_0}\}}\right) \\
&= \mathbb{E}_1\left(H_t 1_{\{t<C_{T_0}\}} \mathbb{E}_1\left(\left(\frac{X_{B_{t+s}}}{X_{B_t}}\right)^{i\lambda} 1_{\{t+s<C_{T_0}\}} \middle| \mathcal{G}_{B_t}\right)\right) \\
&= \mathbb{E}_1\left(H_t 1_{\{t<C_{T_0}\}} \mathbb{E}_{X_{B_t}}\left(\left(\frac{X_{B_s}}{X_0}\right)^{i\lambda} 1_{\{s<C_{T_0}\}}\right)\right) \\
&= \mathbb{E}\big(H_t 1_{\{t<\zeta\}}\big)\, \mathbb{E}\big(\exp\{i\lambda\xi_s\} 1_{\{s<\zeta\}}\big)\,,
\end{aligned}$$

where we used the strong Markov property to pass from the second into the third line, and in the final identity that ξ does not depend on X_0. The result follows.

Here we would like to point out that the same method of proof can be extended to show a bijection between self-similar Markov processes in dimension $d \geq 1$, to Markov additive processes, that will be stated in the final section of these lecture notes. The only difference is that in the construction of ξ a dependence on the starting point of X, say X_0, appears, because it will depend on the angle of X_0, $X_0/|X_0|$.

4 Stable processes

We give some explicit calculations for stable processes. By a (one-dimensional strictly) stable process, we mean a Lévy process with the scaling property: for all $c > 0$,

$$(cX_{tc^{-\alpha}}, t \geq 0) \stackrel{\mathcal{L}}{=} (X_t, t \geq 0), \quad X_0 = 0, \tag{4.1}$$

for some $\alpha \in (0, 2]$, in which case we call X an α-stable process. The characteristic exponent ψ should satisfy

$$\psi(k\lambda) = k^\alpha \psi(\lambda), \qquad \lambda \in \mathbb{R},\ k > 0, \tag{4.2}$$

$$\mathbb{E}\big(e^{i\lambda X_t}\big) = \exp\{-t\psi(\lambda)\}, \qquad \lambda \in \mathbb{R}. \tag{4.3}$$

If $\alpha = 2$, then

$$\psi(\lambda) = \frac{\sigma^2\lambda^2}{2}, \quad \lambda \in \mathbb{R}; \tag{4.4}$$

X is a Brownian motion, and so it is continuous. If $\alpha \in (0,2)$, then X has no continuous part.

If $\alpha \in (0,1) \cup (1,2)$, then

$$\psi(\lambda) = c\,|\lambda|^{\alpha}\Big(1 - i\beta\,\mathrm{sgn}(\lambda)\tan\Big(\frac{\pi\alpha}{2}\Big)\Big), \qquad \lambda \in \mathbb{R} \tag{4.5}$$

and X has a Lévy measure

$$\Pi(dx) = \begin{cases} \frac{c_+ dx}{|x|^{1+\alpha}}, & x > 0, \\ \frac{c_- dx}{|x|^{1+\alpha}}, & x < 0, \end{cases} \tag{4.6}$$

for some constants $c_+, c_- \geq 0$ with $c_+ + c_- > 0$, where $\beta = \frac{c_+ - c_-}{c_+ + c_-}$ is the *skewness parameter* and

$$c = -(c_+ + c_-)\Gamma(-\alpha)\cos\Big(\frac{\pi\alpha}{2}\Big) \quad (> 0). \tag{4.7}$$

The *positivity parameter* $\rho = \mathbb{P}(X_1 > 0)$ is given as

$$\rho = \frac{1}{2} + \frac{1}{\alpha\pi}\arctan\Big(\beta\tan\Big(\frac{\pi\alpha}{2}\Big)\Big), \tag{4.8}$$

which ranges $[0,1]$ for $\alpha \in (0,1)$ and $\big[1 - \frac{1}{\alpha}, \frac{1}{\alpha}\big]$ for $\alpha \in (1,2)$.

If $\alpha = 1$, then

$$\psi(\lambda) = c\,|\lambda| + i\eta\lambda, \quad \lambda \in \mathbb{R} \tag{4.9}$$

for some constants $c \geq 0$ and $\eta \in \mathbb{R}$, and X has a Lévy measure

$$\Pi(dx) = \frac{c dx}{\pi\,|x|^2}. \tag{4.10}$$

In what follows, we only consider the case $c > 0$ and $\eta = 0$, i.e.,

$$\psi(\lambda) = c\,|\lambda|\,, \quad \lambda \in \mathbb{R}; \tag{4.11}$$

consequently the formulae (4.6), (4.7), and (4.8) are still valid for

$$c_+ = c_- = \frac{c}{\pi}, \beta = 0 \text{ and } \rho = \frac{1}{2}. \tag{4.12}$$

4.1 Killed process

Let $(Y_t, t \geq 0)$ be X killed at the first passage time below 0:

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}, \tag{4.13}$$

$$Y_t = \begin{cases} X_t, & t < \tau_0^-, \\ 0, & t \geq \tau_0^-. \end{cases} \tag{4.14}$$

Recall that X issued from x has the same law as $x+X$ where X is issued from 0. For $\alpha \in (0,2)$, we have

$$X_{\tau_0^- -} = \lim_{t\uparrow\tau_0^-} X_t > 0 \quad \Leftrightarrow \quad \Pi(-\infty,0)>0 \quad \Leftrightarrow \quad c_- > 0. \tag{4.15}$$

Theorem 4.1. *We suppose $\alpha \in (0,2)$. Let ξ^* be the Lévy process associated to Y via Lamperti's transform ($Y = e^{\xi^*_{\tau(\cdot)}}$). Then ξ^* has the characteristics given as follows:*

(i) the lifetime $\zeta = \inf\{t>0 : \xi^*_t = -\infty\}$ follows an exponential distribution of parameter $q = \frac{c_-}{\alpha}$. ($\zeta = \infty$ a.s. if $c_- = 0$)

(ii) the Gaussian term of ξ^* is zero.

(iii) Lévy measure of ξ^* is given as

$$\frac{\Pi^*(dy)}{dy} = \frac{c_+ e^y}{(e^y-1)^{1+\alpha}} 1_{\{y>0\}} + \frac{c_- e^y}{|e^y-1|^{1+\alpha}} 1_{\{y<0\}}. \tag{4.16}$$

(iv) If c_+ and c_- are taken as

$$c_+ = \Gamma(1+\alpha)\frac{\sin(\alpha\pi\rho)}{\pi}, \quad c_- = \Gamma(1+\alpha)\frac{\sin(\alpha\pi(1-\rho))}{\pi}, \tag{4.17}$$

then

$$\mathbb{E}\big(e^{i\lambda\xi^*_1} 1_{\{1<\zeta\}}\big) = \exp\{-\psi_{\xi^*}(\lambda)\}, \tag{4.18}$$

$$\psi_{\xi^*}(\lambda) = \frac{\Gamma(\alpha - i\lambda)}{\Gamma(\alpha(1-\rho)-i\lambda)} \times \frac{\Gamma(1+i\lambda)}{\Gamma(1-\alpha(1-\rho)+i\lambda)}, \qquad \lambda \in \mathbb{R}. \tag{4.19}$$

(This is one example of the Wiener–Hopf factorization.)

Proof. The proof of (ii) can be obtained via the creeping argument (Dee, e.g., [27, Theorem 7.11]). If X has positive jumps, then X does not creep upward nor does ξ^*, which shows ξ^* has no Gaussian part.

Let us prove (iii). For that end we recall that the jumps of ξ^* form a Poisson point process whose intensity will be denoted by $\Pi^*(dx)$. So, for $B \in \mathcal{B}((0,\infty)\times(\mathbb{R}\backslash\{0\}))$,

$$\#\big\{t>0 : (t, \xi^*_t - \xi^*_{t-}) \in B\big\} \tag{4.20}$$

$$\sim \text{Poisson distribution with parameter} \int_0^\infty ds \int_{\mathbb{R}\backslash\{0\}} \Pi^*(dx) 1_{\{(s,x)\in B\}}. \tag{4.21}$$

A key property of Poisson point processes is the so-called Compensation formula, that we recall for ease of reference.

Theorem 4.2 (Compensation formula)**.** *Let $(Z_t, t \geq 0)$ be a Lévy process with a Lévy measure Π_Z. Then, for any non-negative predictable process $f(t,x)(\cdot)$ with $f(t,0)(\cdot) = 0$,*

$$\mathbb{E}\left(\sum_{0<t<\zeta} f(t, Z_t - Z_{t-}) \right) = \mathbb{E}\left(\int_0^\zeta dt \int_{\mathbb{R}\setminus\{0\}} \Pi(dx) f(t,x) \right), \tag{4.22}$$

holds for $\Pi = \Pi_Z$, where ζ denotes the lifetime of Z. Conversely, if a measure Π satisfies (4.22) for all non-negative measurable function $f(t,x)$ with $f(t,0) = 0$, then $\Pi = \Pi_Z$.

Recall that $B_t = \inf\{s > 0 : \int_0^s Y_u^{-\alpha} du > t\}$, that $\xi_t^* = \log \frac{Y_{B_t}}{Y_0}$ and that τ_0^- is the lifetime of Y.

$$\mathbb{E}\left(\sum_{0<t<\zeta} f(t, \xi_t^* - \xi_{t-}^*) \right) = \mathbb{E}\left(\sum_{0<t<\zeta} f\left(t, \log \frac{Y_{B_t}}{Y_{B_t-}}\right) 1_{\{Y_{B_t} \neq Y_{B_t-}, B_t < \tau_0^-\}} \right) \tag{4.23}$$

$$= \mathbb{E}\left(\sum_{0<t<\tau_0^-} f\left(\int_0^t Y_s^{-\alpha} ds, \log \frac{Y_t}{Y_{t-}} \right) 1_{\{Y_t \neq Y_{t-}, Y_t > 0\}} \right) \tag{4.24}$$

$$= \mathbb{E}\left(\sum_{0<t<\tau_0^-} f\left(\int_0^t X_s^{-\alpha} ds, \log \frac{X_t}{X_{t-}} \right) 1_{\{X_t \neq X_{t-}, X_t > 0\}} \right). \tag{4.25}$$

By the compensation formula for the stable process X, this is equal to

$$\mathbb{E}\left(\int_0^{\tau_0^-} dt \int_{\mathbb{R}\setminus\{0\}} \Pi(dy) f\left(\int_0^t X_s^{-\alpha} ds, \log\left(1 + \frac{y}{X_{t-}}\right)\right) 1_{\{X_{t-}+y>0\}} \right) = \mathrm{I} + \mathrm{II} \tag{4.26}$$

where I and II are (4.26) with $1_{\{y>0\}}$ and $1_{\{y<0\}}$ being multiplied in the integrand, respectively. For I, we have

$$\mathrm{I} = \mathbb{E}\left(\int_0^{\tau_0^-} dt \int_0^\infty \frac{c_+ dy}{|y|^{1+\alpha}} f\left(\int_0^t X_s^{-\alpha} ds, \log\left(1 + \frac{y}{X_t}\right)\right) 1_{\{y+X_t>0\}} \right) \tag{4.27}$$

$$= \mathbb{E}\left(\int_0^{\tau_0^-} dt X_t^{-\alpha} \int_0^\infty \frac{c_+ dz}{(e^z - 1)^{1+\alpha}} e^z f\left(\int_0^t X_s^{-\alpha} ds, z \right) \right) \tag{4.28}$$

$$= \mathbb{E}\left(\int_0^\zeta dv \int_0^\infty dz \frac{c_+}{(e^z - 1)^{1+\alpha}} e^z f(v,z) \right), \tag{4.29}$$

where we used the change of variables $z = \log\left(1 + \frac{y}{X_t}\right)$, $dy = X_t e^z dz$ and $v = \int_0^t X_u^{-\alpha} du$, $dv = X_t^{-\alpha} dt$, together with the equality $\int_0^{\tau_0^-} X_u^{-\alpha} du = \zeta$. We can make a similar computation for II. By the compensation formula for the Lévy process ξ^*, we obtain (iii).

Let us prove (i). By the compensation formula for X, and using that on the event where X passes below zero by a jump, the time τ_0^- is the unique instant, t, at which $X_s > 0$ for all $s < t$ and $X_t < 0$, we get that for any measurable function $f : (0,\infty) \to (0,\infty)$, we have

$$\mathbb{E}_x\Big(f(\tau_0^-)1_{\{X_{\tau_0^-}<0\}}\Big) = \mathbb{E}_x\Bigg(\sum_{0<t<\zeta} f(t)1_{\{t<\tau_0^-, X_t - X_{t-} < -X_{t-}\}}\Bigg) \tag{4.30}$$

$$= \mathbb{E}_x\bigg(\int_0^\zeta dt \int_{\mathbb{R}\setminus\{0\}} \Pi(dy) f(t) 1_{\{t<\tau_0^-, y<-X_{t-}\}}\bigg) \tag{4.31}$$

$$= \int_0^\infty f(t)\mathbb{E}_x\big(\Pi(-\infty,-X_t), t < \tau_0^-\big)\,. \tag{4.32}$$

Notice that the final equality is a consequence of the fact that the discontinuities of X are countable. Recall that because X is a stable process, it does not creep downward, i.e. $\mathbb{P}_x\Big(X_{\tau_0^-} < 0, \tau_0^- < \infty\Big) = 1$, for all $x > 0$, see e.g. [4]. Thus taking $f \equiv 1$, we have

$$1 = \int_0^\infty \mathbb{E}_x\big(\Pi(-\infty,-X_t), t < \tau_0^-\big)\,dt = \mathbb{E}_x\Bigg(\int_0^{\tau_0^-} \Pi(-\infty,-X_t)dt\Bigg)\,. \tag{4.33}$$

Note that $\Pi(-\infty,-x) = \frac{c_-}{\alpha}x^{-\alpha}$, for $x > 0$. Since $\zeta = \inf\{t > 0 : \xi_t^* = -\infty\} = \int_0^{\tau_0^-} X_s^{-\alpha}ds$, we have

$$1 = \frac{c_-}{\alpha}\mathbb{E}_x(\zeta) = \frac{c_-}{\alpha}\frac{1}{q}, \tag{4.34}$$

which shows $q = \frac{c_-}{\alpha}$. □

4.2 Two conditional processes

Two more processes related to the stable processes are:

(i) $Y^\uparrow$: X conditioned to stay positive.

(ii) $Y^\downarrow$: X conditioned to reach 0 continuously.

We will see that these processes are self-similar and they are obtained as the exponential of a Lévy process, $\xi^\uparrow$ and $\xi^\downarrow$, respectively, time changed,

$$Y^\uparrow = \mathrm{LT}(\xi^\uparrow), \qquad Y^\downarrow = \mathrm{LT}(\xi^\downarrow), \tag{4.35}$$

i.e., $\xi^\uparrow$ and $\xi^\downarrow$ are the Lévy processes associated to $Y^\uparrow$ and $Y^\downarrow$ via Lamperti's transform, respectively. The law of $Y^\uparrow$ is that obtained by conditioning

$$\mathbb{E}_x^\uparrow(F(X_s, s \le t)) = \lim_{u\uparrow\infty} \mathbb{E}_x\big(F(X_s, s\le t)|\tau_0^- > t+u\big) \tag{4.36}$$

for all non-negative bounded measurable functional F. See Chaumont–Doney(2005)[16].

Theorem 4.3. *For $\overline{\rho} = 1 - \rho$,*

$$\mathbb{P}^{\uparrow}_x|_{\mathcal{F}_t} = \frac{X_t^{\alpha\overline{\rho}}}{x^{\alpha\overline{\rho}}} \cdot \mathbb{P}_x|_{\mathcal{F}_t \cap \{t < \tau_0^-\}}. \tag{4.37}$$

This uses that

$$\mathbb{P}_x\big(\tau_0^- > t\big) \underset{t\uparrow\infty}{\sim} c_{\alpha\overline{\rho}} \frac{x^{\alpha\overline{\rho}}}{t^{\overline{\rho}}} \tag{4.38}$$

which can be read in [4]. As a consequence of fluctuation theory, the absolute continuity relation in (4.37) holds true for stopping times due to the optional stopping theorem, in particular for the random time $B_t = \inf\big\{s > 0 : \int_0^s X_u^{-\alpha} du > t\big\}$. Hence

$$\mathbb{P}^{\uparrow}_x|_{\mathcal{F}_{B_t}} = \frac{X_{B_t}^{\alpha\overline{\rho}}}{x^{\alpha\overline{\rho}}} \cdot \mathbb{P}_x|_{\mathcal{F}_{B_t} \cap \{B_t < \tau_0^-\}} = e^{\alpha\overline{\rho}\xi_t^*} \cdot \mathbb{P}^*_x|_{\mathcal{F}_{B_t}} \tag{4.39}$$

(See Chaumont–Doney(2005)[16].)

If Y is the pssMp obtained by killing a stable process on $(-\infty, 0)$, then $Y = \mathrm{LT}(\xi^*)$. ξ^* has a Laplace exponent

$$\psi_{\xi^*}(\lambda) = \frac{\Gamma(\alpha\rho - i\lambda)}{\Gamma(-i\lambda)} \frac{\Gamma(1 + \alpha\overline{\rho} + i\lambda)}{\Gamma(1 + i\lambda)}, \qquad \lambda \in \mathbb{R}. \tag{4.40}$$

See e.g. [27]. The stable process conditioned to stay positive, which we denote by $Y^{\uparrow}$, is a pssMp, this can be easily obtained from (4.39), and recall

$$Y^{\uparrow} = \mathrm{LT}(\xi^{\uparrow}). \tag{4.41}$$

We denote by $P^{\uparrow}$ the law of $\xi^{\uparrow}$ and denote by P^* the law of ξ^*. The identity (4.39) reads

$$P^{\uparrow}|_{\mathcal{F}_t} = \exp\{\alpha\overline{\rho}\xi_t^*\} P^*|_{\mathcal{F}_t \cap \{t < \zeta\}}. \tag{4.42}$$

By analytic continuation, the latter implies the following identitites

$$e^{-t\psi^{\uparrow}(\lambda)} = E^{\uparrow}\big(e^{i\lambda\xi_t}\big) = \mathbb{E}\big(e^{i\lambda\xi_t} e^{\alpha\overline{\rho}\xi_t} 1_{\{t<\zeta\}}\big) = \mathbb{E}\big(e^{i(\lambda - i\alpha\overline{\rho})\xi_t} 1_{\{t<\zeta\}}\big) \tag{4.43}$$

$$= \exp\{-t\psi_{\xi^*}(\lambda - i\alpha\overline{\rho})\}, \quad \lambda \in \mathbb{R}. \tag{4.44}$$

Said otherwise,

$$\psi^{\uparrow}(\lambda) = \psi_{\xi^*}(\lambda - i\alpha\overline{\rho}), \qquad \lambda \in \mathbb{R}. \tag{4.45}$$

So, we have

$$\psi^{\uparrow}(\lambda) = \frac{\Gamma(\alpha - i\lambda)}{\Gamma(\alpha\overline{\rho} - i\lambda)} \frac{\Gamma(1 + i\lambda)}{\Gamma(1 - \alpha\overline{\rho} + i\lambda)}. \tag{4.46}$$

Let $\xi^{\downarrow}$ be the Lévy process associated to $Y^{\downarrow}$, which is a stable process conditioned to hit 0 continuously.

The characteristic exponents, ψ^*, $\psi^\uparrow$ and $\psi^\downarrow$ have the form

$$\psi^\downarrow(x) = \frac{\Gamma(\alpha\rho + 1 - i\lambda)}{\Gamma(1 - i\lambda)} \frac{\Gamma(\alpha\overline{\rho} + i\lambda)}{\Gamma(i\lambda)}, \qquad \lambda \in \mathbb{R}. \tag{4.47}$$

For $\beta \leq 1$, $\gamma \in (0,1)$, $\widehat{\beta} \geq 0$ and $\widehat{\gamma} \in (0,1)$,

$$\psi(\theta) = \frac{\Gamma(1 - \beta + \gamma - i\theta)}{\Gamma(1 - \beta - i\theta)} \frac{\Gamma(\widehat{\beta} + \widehat{\gamma} + i\theta)}{\Gamma(\widehat{\beta} + i\theta)}. \tag{4.48}$$

The latter is the characteristic exponent of a so-called hypergeometric Lévy process, that we may denote here by $\widetilde{\xi}$ (See, [25]). For the function

$$\theta \mapsto \frac{\Gamma(1 - \beta + \gamma - i\theta)}{\Gamma(1 - \beta - i\theta)} = \kappa(-i\theta), \tag{4.49}$$

κ is the Laplace exponent of the upward ladder height process associated to $\widetilde{\xi}$, while for the function

$$\theta \mapsto \frac{\Gamma(\widehat{\beta} + \widehat{\gamma} + i\theta)}{\Gamma(\widehat{\beta} + i\theta)} = \widehat{\kappa}(i\theta), \tag{4.50}$$

$\widehat{\kappa}$ is the Laplace exponent of the downward ladder height process associated to $\widetilde{\xi}$.

There are other examples of hypergeometric Lévy processes related to stable process via Lamperti's transform. As for instance, let X be symmetric stable process. We let $Y = |X| = \mathrm{LT}(\widehat{\xi})$. Then $\widehat{\xi}$ is an hypergeometric Lévy process.

Let X be a stable process and $Y = X_{c_t}$ where $c_t = \inf\left\{s > 0 : \int_0^s 1_{\{X_u \geq 0\}} du > t\right\}$. So, that Y is the process that is obtained by removing the negative part of the stable process X. It is an interesting exercise to verify that Y preserves both the scaling property and the strong Markov property. So it is a pssMp. We let $Y = \mathrm{LT}(\xi')$. Furthermore, a small analysis of the paths of X and Y allows to realize that Y is obtained by gluing independent copies of Y^* at random positions determined by the position reached by X when returning at $(0,\infty)$. So, one may naturally ask how ξ' is related to ξ^*. It is not hard to realize that ξ' should have the same law as $\xi + z$ where z is an independent compound Poisson process with rate $\frac{c_-}{\alpha}$ and some jump distribution F, this has been proved by Kyprianou–Pardo–Watson(2014)[28]).

Exercise 4.4. Prove that Y is a pssMp.

5 Entrance law of pssMp

If X is a pssMp with $X = \mathrm{LT}(\xi)$, then X dies at 0. Lamperti(1972)[30] asked whether we can allow entrance of X from 0 into $(0,\infty)$ so that X does not die at 0. There are well known examples of self-similar Markov processes that can be started from 0, as for instance Bessel processes and stable processes conditioned to stay positive. Notice that if X returns to 0, by the scaling and the strong Markov property one can verify that 0 should be a recurrent and a regular state (e.g., the reflected Brownian motion). When $X = \mathrm{LT}(\xi)$ can be started from 0 and X does not return to 0 (i.e., $T_0 = \infty$), the question is whether there exists a probability measure $\mathbb{P}_{0+}$ that can be obtained $\mathbb{P}_x \underset{x\downarrow 0}{=} \mathbb{P}_{0+}$ in the sense of weak convergence w.r.t. the Skorohod topology. The answer was obtained by Bertoin–Caballero(2002)[5], Bertoin–Yor(2002)[8], Caballero–Chaumont(2006)[11] and Chaumont–Kyprianou–Pardo–Rivero(2012)[17]. $\mathbb{P}_{0+}$ exists and it is non-degenerate if $[\mathbb{E}(|\xi_1|) < \infty$ and $\mathbb{E}(\xi_1) > 0]$ or $[\mathbb{E}(\xi_1) = 0$ and the upward ladder height process of ξ has a finite mean]. $\mathbb{P}_{0+}$ is constructed by building a process ξ^* on $(-\infty, \infty)$ (See Bertoin–Savov(2011)[7] and the papers mentioned above).

When $\mathbb{E}(|\xi_1|) < \infty$ and $\mathbb{E}(\xi_1) > 0$, then $\mathbb{P}_{0+}$ can be written as

$$\mathbb{P}_{0+}(f_1(X_{t_1}), \dots, f_n(X_{t_n}))) = \int \mu_t(dx_1) f_1(x_1) \mathbb{E}_{x_1}(f_2(X_{t_2-t_1}, \dots, f_n(X_{t_n-t_1}))), \tag{5.1}$$

where $(\mu_t(dx), t > 0)$ is an entrance law for X, meaning that

$$\int \mu_t(dx) \mathbb{E}_x(f(X_s)) = \int \mu_{t+s}(dx) f(x) \tag{5.2}$$

and

$$\mathbb{E}_{0+}(f(X_t)) = \int \mu_t(dx) f(x) = \frac{1}{\alpha \mathbb{E}(\xi_1)} \mathbb{E}\left(f\left(\frac{t^{\frac{1}{\alpha}}}{\widehat{I}^{\frac{1}{\alpha}}} \right) \frac{1}{\widehat{I}} \right) \tag{5.3}$$

where $\widehat{I} = \int_0^\infty e^{-\alpha \xi_s} ds$ with ξ such that $X = \mathrm{LT}(\xi)$. For a general formula see [17] and .

For the answer to the question *What are the positive α–self-similar Markov processes $\widetilde{X}$ which behave like $(X, \mathbb{P})$ up to the first hitting time of 0 for $\widetilde{X}$ and such that 0 is a regular and recurrent state?* We refer to [36, 37, 39] and [20]. A process that has this characteristics is usually called a recurrent extension of the process $(X, \mathbb{P})$.

6 Exponential functionals

In the theory of pssMp, exponential functionals are found everywhere; first hitting times, entrance laws, asymptotic behaviour of X, first passage above levels, quasi-stationary distributions of X, etc. Laws of iterated logarithm are given in terms of exponential functionals. Exponential functionals also appear in finance, risk theory, time series, statistical physics, etc. See the survey by Bertoin and Yor(2005)[9], where many of our claims can be found.

We need to develop tools to study exponential functionals of Lévy processes: to get explicit law, estimates for the distribution and distributional properties (infinite divisibility).

Let ξ be a Lévy process either with a finite lifetime ($q > 0$), or with $q = 0$ and $\xi_t \underset{t\to\infty}{\to} -\infty$. These conditions are necessary and sufficient for $\int_0^\infty e^{\xi_s} ds < \infty$ a.s.

Here we will describe exponential functionals using various approaches. Exponential functionals as perpetuities:

$$I = \int_0^\infty e^{\xi_s} ds. \tag{6.1}$$

Let T be a stopping time of ξ and decompose

$$I = \int_0^T e^{\xi_s} ds + e^{\xi_T} 1_{\{T<\infty\}} \int_0^\infty e^{\xi_{s+T}-\xi_T} ds. \tag{6.2}$$

By the strong Markov property of ξ, $Q := \int_0^T e^{\xi_s} ds$ and $M := e^{\xi_T} 1_{\{T<\infty\}}$ are independent of $\int_0^\infty e^{\xi_{s+T}-\xi_T} ds \overset{\mathcal{L}}{=} I$. So $I \overset{\mathcal{L}}{=} Q + M\widetilde{I}$ where $\widetilde{I}$ is a copy of I independent of (Q, M). The theory of perpetuities of Goldie(1991)[21] and others, is available for the study of I.

We have freedom to choose T so the decomposition mentioned above is very useful.

For the study Yaglom limits of pssMp, useful is to choose $T = \tau(t) = \inf\{s > 0 : \int_0^s e^{\xi_u} du > t\}$ for some $t > 0$. Then we have

$$I \overset{\mathcal{L}}{=} t + e^{\xi_{\tau(t)}} 1_{\{\tau(t)<\infty\}} \widetilde{I} = t + X_t \widetilde{I}, \qquad X = \mathrm{LT}(\xi). \tag{6.3}$$

Using this fact, we can see that the following conditions are equivalent:

(i) There exists a function $g : (0,\infty) \to (0,\infty)$ and a non-degenerate measure μ such that

$$\mathbb{P}_1\left(\frac{X_t}{g(t)} \in dy \middle| t < T_0\right) \to \mu(dy), \tag{6.4}$$

in this case, we will say that μ is a Yaglom limit for X conditioned to stay alive.

(ii) There exists a measure $\widetilde{\Lambda}(dy)$ such that

$$\mathbb{P}\left(\frac{I-t}{g(t)} \in dy \middle| t < I\right) \to \widetilde{\Lambda}(dy). \tag{6.5}$$

(iii) I belongs to the maximum domain of attraction of an extremal distribution, i.e., there exist sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, such that if $(I_j, j \geq 1)$ are i.i.d. with $I_j \overset{\mathcal{L}}{=} I$,

$$\frac{\max\{I_1, \ldots, I_n\} - a_n}{b_n} \underset{n\uparrow\infty}{\to} \text{a non-degenerate random variable.} \tag{6.6}$$

If X satisfies one (and hence all) of these three conditions, then $\widetilde{\Lambda}$ can only be either an exponential, beta, or Pareto distribution. See Haas and Rivero(2012)[23].

Suppose ξ is a spectrally positive Lévy process, i.e., $\Pi(-\infty,0)=0$. Choose $T=\tau_y^-=\inf\{t>0:\xi_t<y\}$. Since $\xi_{\tau_y^-}=y$, if $\tau_y^-<\infty$, we have

$$I=\int_0^{\tau_y^-} e^{\xi_s}ds+e^y 1_{\{\tau_y^-<\infty\}}\widetilde{I}. \tag{6.7}$$

If $q=0$, then $\xi_t\to-\infty$, as $t\to\infty$, hence $\tau_y^-<\infty$, and

$$I=\int_0^{\tau_y^-} e^{\xi_s}ds+e^y\widetilde{I}. \tag{6.8}$$

Hence I is self-decomposable and in particular infinitely divisible with Laplace transform

$$\mathbb{E}\big(e^{-\lambda I}\big)=\exp\bigg\{-\int_0^\infty(1-e^{-\lambda x})k(x)\frac{dx}{x}\bigg\} \tag{6.9}$$

(See Sato(2013)[34]), where k is a non-increasing function given as

$$k(x)=\Pi_Y(x,\infty),\qquad x>\infty \tag{6.10}$$

with Π_Y being the Lévy measure of a Lévy process Y such that

$$I\overset{\mathcal{L}}{=}\int_0^\infty e^{-s}dY_s. \tag{6.11}$$

Note that every self-decomposable random variable can be written this way. k determines the behaviour of $\mathbb{P}(I\in dt)$. In the present setting, it can be verified that Y is a subordinator such that

$$Y_t=\sum_{s\le t}\Delta Y_s, \tag{6.12}$$

where $\Delta Y_t=\int_{\tau_{t-}^-}^{\tau_t^-}e^{\xi_s}ds$, see e.g. [38].

Carmona–Petit–Yor(1994)[12], (1997)[13], (2001)[14] where they study several properties of I. They proved that there is always a density

$$\mathbb{P}(I\in dy)=p(y)dy. \tag{6.13}$$

$p\in C^\infty$ and p solves the equation:

$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2p(x))+\bigg(\Big(\frac{\sigma^2}{2}+a\Big)x+1\bigg)p(x)k(x) \tag{6.14}$$

$$=\int_x^\infty\Pi\Big(-\infty,\log\frac{u}{x}\Big)p(u)du+\int_0^x\Pi\Big(\log\frac{x}{u},\infty\Big)p(u)du,\quad x>0. \tag{6.15}$$

When $\Pi=0$, this equation can be used to verify that for $b>0$ and $\sigma>0$,

$$\int_0^\infty e^{\sigma B_t-bt}dt\overset{\mathcal{L}}{=}\frac{2}{\sigma^2\gamma_{\frac{2b}{\sigma^2}}}, \tag{6.16}$$

where γ_λ is a Gamma random variable of parameter λ, $\lambda > 0$.

Carmona–Petit–Yor(1994)[12] and Bertoin–Yor(2002)[8], proved the following moment formula for exponential functionals of subordinators (see Theorem 2 of Bertoin–Yor(2005)[9]). When $\xi = -\sigma$ where σ is a subordinator and $I = \int_0^\infty e^{-\sigma_s} ds$, then

$$\infty > \mathbb{E}(I^n) = \prod_{k=0}^{n} \frac{k}{\phi(k)}, \qquad n \geq 0, \tag{6.17}$$

where ϕ is the Laplace exponent of σ, so

$$-\log \mathbb{E}\big(e^{-\lambda\sigma_1} 1_{\{1<\zeta\}}\big) = \phi(\lambda) = q + d\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx). \tag{6.18}$$

I is determined by its moments. There exists a random variable R_ϕ such that if I is independent from R_ϕ, then

$$IR_\phi \stackrel{\mathcal{L}}{=} \mathbf{e}_1 \tag{6.19}$$

and the moments of R_ϕ are given by

$$\mathbb{E}(R_\phi^n) = \prod_{k=1}^{n} \phi(k). \tag{6.20}$$

A remarkable fact is that $\log R_\phi$ is infinitely divisible and spectrally positive, this and fact and further properties about it can be found in the paper by Alili–Jedidi–Rivero(2014)[1] and the reference therein.

Example 6.1. Let σ be the subordinator with $q = \frac{1}{\Pi(1-\alpha)}$, $d = 0$ and

$$\Pi(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-x}}{(1-e^{-x})^{1+\alpha}} dx. \tag{6.21}$$

Let X be the pssMp such that X is obtained applying the Lamperti transformation to $(\alpha\sigma)$. Then $X^{\frac{1}{\alpha}} \stackrel{\mathcal{L}}{=} -Y$ for some stable subordinator Y of parameter α. We denote

$$T_0^{(x)} = \inf\{t > 0, \left(X_t^{(x)}\right)^{\frac{1}{\alpha}} < 0\} = \inf\{t > 0, Y_t > x\}. \tag{6.22}$$

It is known that if $x = 1$, then $T_0^{(1)} = S_\alpha^{-\alpha} \stackrel{\mathcal{L}}{=} {Y_1}^\alpha$ where S_α is a non-negative random variable with parameter α. Indeed, this follows from

$$\begin{aligned}
\mathbb{P}(T_0^{(1)} > t) &= \mathbb{P}\left(\left(X_t^{(1)}\right)^{\frac{1}{\alpha}} > 0\right) \\
&= \mathbb{P}(Y_t < 1) \\
&= \mathbb{P}(t^{\frac{1}{\alpha}} Y_1 < 1) \\
&= \mathbb{P}(t < Y_1^{-\alpha}).
\end{aligned}$$

Said otherwise, $T_0^{(1)}$ has Mittag Leffler distribution, and hence its moments are given by

$$\mathbb{E}\Big(\Big(T_0^{(1)}\Big)^n\Big) = \frac{n!}{\Gamma(1+\alpha n)}. \tag{6.23}$$

It has been proved by Shanbhag–Sreehari(1977)[35], that

$$S_\alpha^{-\alpha}\widetilde{\mathbf{e}}^\alpha \stackrel{\mathcal{L}}{=} \mathbf{e}, \tag{6.24}$$

where $\mathbf{e}$ and $\widetilde{\mathbf{e}}$ are exponential random variables and $\widetilde{\mathbf{e}}$ is independent of S_α. This fact is easy to verify by calculating the moments of these r.v. and using the fact that the exponential random variable is moment determinate.

The factorisations of random variables using exponential functionals or of exponential functionals using other r.v. is a topic that has been the point of many research works. As for the subordinator case, this is motivated by the recursive form of its moments. More precisely, the following formula holds. Let C be the set

$$C = \{\lambda \in \mathbb{R} : \mathbb{E}\big(e^{\lambda\xi_1}1_{\{a<\zeta\}}\big) < \infty\}. \tag{6.25}$$

For any $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) \in C \cap (0,\infty)$, we have

$$\mathbb{E}\big(I^\beta\big) = \frac{\beta}{\psi_\xi(-i\beta)}\mathbb{E}\big(I^{\beta-1}\big) \tag{6.26}$$

This can be found in [33]. Recall that the Wiener–Hoph factorization of ξ establishes that

$$\psi_\xi(-i\beta) = \kappa_H(i\beta)\kappa_{\widehat{H}}(-i\beta). \tag{6.27}$$

We have then that

$$\mathbb{E}\big(I^\beta\big) = \frac{\beta}{\kappa_H(i\beta)}\frac{1}{\kappa_{\widehat{H}}(-i\beta)}\mathbb{E}\big(I^{\beta-1}\big) \tag{6.28}$$

where H is the upward ladder height process, $\widehat{H}$ is the downward ladder height process and κ_H and $\kappa_{\widehat{H}}$ are these Laplace exponents, respectively. We also have

$$\mathbb{E}\Big(I_{-\widehat{H}}^\beta\Big) = \frac{\beta}{\kappa_H(i\beta)}\mathbb{E}\Big(I_{-\widehat{H}}^{\beta-1}\Big) \tag{6.29}$$

where $I_{-\widehat{H}} = \int_0^\infty e^{-\widehat{H}_s}ds$. The former and later identities allow to guess that the exponential functional $I_{-\widehat{H}}$ should be involved in a factorization of I. This remarkable fact was observed by Pardo–Patie–Savov(2012)[33] by proving that

$$I \stackrel{\mathcal{L}}{=} I_{-\widehat{H}}\mathcal{J}_H \tag{6.30}$$

for some independent random variable $\mathcal{J}_{\widehat{H}}$ which depends on $\kappa_{\widehat{H}}$. (See Pardo–Patie–Savov(2012)[33].) Furthermore, one has the identity

$$\mathbb{P}(\mathcal{J}_H \in dy) = cy\mathbb{P}(R_H \in dy) \tag{6.31}$$

where R_H is a random variable, independent of I_{-H}, whose moments are given by [3], and it satisfies that

$$I_{-H}R_H \stackrel{\mathcal{L}}{=} \mathbf{e}_1. \tag{6.32}$$

In [3] it has been verified that

$$I \stackrel{\mathcal{L}}{=} e^{S_\infty}\frac{I_{-\widehat{H}}}{R_H} \tag{6.33}$$

where $S_\infty = \sup_{s\geq 0}\xi_s$, and S_∞, $I_{-\widehat{H}}$ and R_H are independent. Using this decomposition, it has been verified in [3], that under some general assumptions

$$\mathbb{P}(I > t) \sim t^{-\alpha}l(t) \Leftrightarrow \mathbb{P}\big(e^{S_\infty} > t\big) \sim t^{-\alpha}\widetilde{l}(t), \tag{6.34}$$

where l and $\widetilde{l}$ are slowly varying functions. Also in [3] it has been shown that under some assumptions

$$\mathbb{P}(I \leq t) \sim \mathbb{P}\big(I_{\widehat{I}} \leq t\big)\,, \tag{6.35}$$

$$\mathbb{P}(I \leq t) \sim t^{\beta}\widehat{l}(t) \tag{6.36}$$

for some $\beta \geq 0$ where $\widehat{l}$ is a slowly varying function.

For further reading, here is a list of some recent studies of pssMp's and exponential functionals.

- M. Barczy and L Döring. On entire moments of self-similar Markov processes. Stoch. Anal. Appl. 31 (2013), no. 2, 191–198.
- L. Chaumont, A. E. Kyprianou and J. C. Pardo. Some explicit identities associated with positive self-similar Markov processes. Stochastic Process. Appl. 119 (2009), no. 3, 980–1000.
- L. Chaumont and J. C. Pardo. The lower envelope of positive self-similar Markov processes. Electron. J. Probab. 11 (2006), no. 49, 1321–1341.
- L. Chaumont and V. Rivero. On some transformations between positive self-similar Markov processes. Stochastic Process. Appl. 117 (2007), no. 12, 1889–1909.
- L. Chaumont, A. E. Kyprianou, J. C. Pardo and V. Rivero. Fluctuation theory and exit systems for positive self-similar Markov processes. Ann. Probab. 40 (2012), no. 1, 245–279.
- A. Kuznetsov. On the distribution of exponential functionals for Lévy processes with jumps of rational transform. Stochastic Process. Appl. 122 (2012), no. 2, 654–663.
- P. Patie. Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration. Bull. Sci. Math. 133 (2009), no. 4, 355–382.

- P. Patie. Infinite divisibility of solutions to some self-similar integro-differential equations and exponential functionals of Lévy processes. (English, French summary) Ann. Inst. Henri Poincaré Probab. Stat. 45 (2009), no. 3, 667–684.
- P. Patie. Law of the absorption time of some positive self-similar Markov processes. Ann. Probab. 40 (2012), no. 2, 765–787.
- P. Patie and M. Savov. Bernstein-gamma functions and exponential functionals of Lévy processes. Electron. J. Probab. 23 (2018), Paper No. 75, 101 pp.

7 Lamperti–Kiu transform and Markov additive processes

Consider a completed, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), (\mathbb{P}_{\varphi,x}, (\varphi, c) \in E \times \mathbb{R}))$; where E is a locally compact space with a countable base, Δ is some isolated state and $E \cup \{\Delta\}$ endowed with its Borel σ-field.

Definition 7.1 (Neveu(1961)[32], Çinlar(1972)[15])**.** A *Markov additive process* (MAP) is an $E \times \mathbb{R}$-valued strong Markov process $\{(J, \xi), \mathbb{P}_{\varphi,x}\}$ with a cemetery state (Δ, ∞) and a lifetime ζ such that

(i) the paths of (J, ξ) are right continuous on $(0, \zeta)$, have left-limits and are quasi-left-continuous on $[0, \zeta)$.

(ii) J is a strong Markov process.

(iii) for any $(\varphi, z) \in E \times \mathbb{R}$, $t, s \geq 0$ and a positive measurable function $f : E \times \mathbb{R} \to \mathbb{R}$,

$$\mathbb{P}_{\varphi,x}(f(J_{t+s}, \xi_{t+s} - \xi_t), t + s < \zeta | \mathcal{F}_t) = \mathbb{P}_{J_t,0}(f(J_s, \xi_s), s < \zeta) 1_{\{t<\zeta\}}. \tag{7.1}$$

Theorem 7.2 (Lamperti(1972)[30], Kiu(1980)[24], Chaumont–Pantí–Rivero(2013)[18], Kuznetsov–Kyprianou–Pardo–Watson(2014)[26], Alili–Chaumont–Graczyk–Zak(2016)[2])**.** *Let X be a $\mathbb{R}^d$-valued strong Markov process having càdlàg paths and being quasi-left-continuous which has the scaling property: there exists an $\alpha > 0$ such that for any $c > 0$*

$$\{(cX_{c^{-\alpha}t}, t \geq 0), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \{(X_t, t \geq 0), \mathbb{P}_{cx}\}, \quad x \in \mathbb{R}^d \tag{7.2}$$

Assume X dies at its first hitting time of 0. *Then the process (J, ξ) defined by*

$$J_t = \frac{X_{\tau(t)}}{\left|X_{\tau(t)}\right|}, \qquad \xi_t = \log\left(\frac{\left|X_{\tau(t)}\right|}{|X_0|}\right), \qquad t \geq 0, \tag{7.3}$$

(ξ_t depends on x only via $\frac{x}{|x|}$) with

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s |X_u|^{-\alpha} \, du > t \right\}, \quad t > 0, \tag{7.4}$$

and $(J, \xi)_{\tau(t)} = (\Delta, \infty)$ if $\tau(t) = \infty$, is a $\mathbb{S}^{d-1} \times \mathbb{R}$-valued Markov additive process.

For further reading, here is a list of some recent studies of ssMp's where the Lamperti–Kiu transforms are utilized.

- L. Alili, L. Chaumont, P. Graczyk and T. Żak. Inversion, duality and Doob h-transforms for self-similar Markov processes. Electron. J. Probab. 22 (2017), Paper No. 20, 18 pp.
- M. E. Caballero, J. C. Pardo and J. L. Pérez. On Lamperti stable processes. Probab. Math. Statist. 30 (2010), no. 1, 1–28.
- S. Dereich, L. Döring and A. E. Kyprianou. Real self-similar processes started from the origin. Ann. Probab. 45 (2017), no. 3, 1952–2003.
- L. Döring, A. E. Kyprianou and P. Weissmann. Stable process conditioned to avoid an interval. arXiv:1802.07223, 2018.
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