KTGU Special Lectures
Blow-up, compactness and (partial) regularity in Partial Differential Equations

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Abstract. The question of whether solutions of Partial Differential Equations (PDEs) are regular or not is central in the field. One of the most famous problems is the existence of smooth solutions to the Navier-Stokes equations in fluid mechanics, or the finite time break down of regularity (millennium problem of the Clay Institute).

The scope of this lecture series is much more modest. Methods based on blow-up and compactness are powerful tools to establish regularity for linear PDEs or partial regularity for nonlinear PDEs. These methods, which originated in the study of the regularity of minimal surfaces in the 60's, have been successfully applied to other subjects: regularity in homogenization, in the calculus of variations or in fluid mechanics. More specifically, the lectures will focus on two topics: (i) uniform estimates in the homogenization of linear elliptic divergence form equations, (ii) epsilon-regularity results for the Navier-Stokes equations. The material presented in the course is well-known to the PDE community since the late 90's. However, the results have been celebrated as breakthroughs and are still inspiring new mathematical developments today, some of which will be outlined.

Summary of the content:
1. Improved regularity in homogenization: compactness methods for uniform Lipschitz regularity, Liouville type theorems for equations with periodic coefficients

2. Epsilon-regularity for Navier-Stokes equations

The lectures are based on works by Avellaneda and Lin (1987, 1989, 1991), Caffarelli, Kohn and Nirenberg (1982), Lin (1998), Ladyzhenskaya and Seregin (1999), and Kukavica (2009).

1 Lectures 1 – 2: Compactness methods in homogenization

1.1 Introduction

This lecture is based on a d-dimensional linear elliptic equation, $d \geq 2$,

$$-\nabla \cdot \mathbf{a}(x)\nabla u \,=\, 0\,, \quad x \in B(0,R) \subset \mathbb{R}^d\,. \tag{1}$$

Here $u = u(x) \in \mathbb{R}$ is the unknown function and $\mathbf{a}(x) \in \mathbb{R}^{d\times d}$ is a given matrix function. We denote by $B(x,R)$ the ball centered at a point $x_0 \in \mathbb{R}^d$ with radius $R \in (0,\infty)$:

$$B(x_0,R) \,=\, \{x \in \mathbb{R}^d \mid |x-x_0| < R\}\,.$$

We are interested in the regularity theory for the equation (1), namely, the local behavior of solutions. Especially, we emphasize the following aspects:

Global vs Local

Global approach: The PDE is considered as an evolution equation with the initial condition. We will obtain the solutions in some Sobolev space, and study the growth of the norms. The regularity will also be studied in terms of, for example, Fourier series of the solutions.

Local approach (this is the approach we will take in this lecture): The PDE is considered completely locally. The goal is to obtain the estimates for the solutions in high regularity norm on a ball in physical space, by assuming that the solutions are controlled in lower regularity norm but on some bigger ball. A typical estimate can be written as

$$\|\nabla u\|_{L^\infty(B(0,\frac{1}{2}))} \le C\|u\|_{L^2(B(0,1))} \,. \tag{2}$$

Polynomials
We determine the building blocks of the regularity theory. In the Taylor expansion formula, the blocks are just the polynomials. We will establish expansion results at the PDE level, which is called the Liouville-type theorems.

Next we underline a few recurrent themes in this lecture.

Localization
We always localize the problem by using test functions, etc.

Multiscale
As can be seen from (2), large scales will control small scales. We prove, for example, a characterization of the Hölder continuity of the solutions to (1) in terms of decay of

$$\fint_{B(0,\rho)} |\nabla u|^2 \,, \tag{3}$$

where we have set for an open set $\Omega \subset \mathbb{R}^d$,

$$\fint_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega} \,. \tag{4}$$

Comparison to polynomials
We compare the solutions to (1) with the building blocks (polynomials) in the regularity theory. For example, we will study the following quantity for $a \in \mathbb{R}$,

$$\fint_{B(0,\rho)} |u-a|^2 \,. \tag{5}$$

Compactness (improvement of flatness)
The basic idea is that if we take some limit of the equation, we will have a new equation for which the regularity properties are better. Examples are the followings:

- Zooming in: Let $\mathbf{a}$ in the equation (1) satisfy $\mathbf{a} \in C^{0,\mu}$ for $\mu \in (0,1)$. Then if we zoom the equation around $0 \in \mathbb{R}^d$, we obtain an equation with a constant coefficient $\mathbf{a}(0)$. The regularity properties are better for the new equation.

- Zooming out: Let $\mathbf{a}(x)$ in the equation (1) be replaced by $\mathbf{a}(\frac{x}{\epsilon})$ for $\epsilon \in (0,1)$. Then if we take the limit $\epsilon \to 0$ under the structure assumption on $\mathbf{a}$ (periodicity, for example), we wil have an equation with some constant coefficient. Then the regularity of solutions is better for the limit equation.
- Convergence to linear equation: If we consider a nonlinear problem and if the nonlinearity is weak for some reason, we can obtain the linear equation in a suitable limit. This is indeed the ϵ-regularity result case, which we will discuss in Lectures 3 – 4.

The idea of the improvement of flatness is originated in the works by Almgren [1] for the regularity of minimal surfaces, and of Evans and Gariepy [6], and Giaquinta [7] for the calculus of variations. Lecture 1 – 2 are based on the papers by Avellaneda and Lin [2, 3].

1.2 Caccioppoli's inequality

We consider an elliptic problem

$$-\nabla \cdot \mathbf{a}(x)\nabla u \,=\, 0\,, \quad x \in B(0,1) \subset \mathbb{R}^d\,. \tag{6}$$

Here $\mathbf{a}(x) = (a_{\alpha\beta}(x))_{\alpha,\beta\in\{1,\dots,d\}} \in \mathbb{R}^{d\times d}$ and $a_{\alpha\beta}(x)$ is measurable for $\alpha, \beta \in \{1,\dots,d\}$. We assume that we have for $\Lambda \in (0,\infty)$ and $L \in (0,\infty)$,

$$\begin{aligned}&\mathbf{a} \text{ is elliptic with a constant } \Lambda\,, \text{namely}\,,\\ &\qquad \mathbf{a}(x)\xi\cdot\xi \geq \Lambda|\xi|^2\,, \quad x\,,\xi \in \mathbb{R}^d\,,\\ &\text{and } \|\mathbf{a}\|_{L^\infty} \leq L\,.\end{aligned}$$

We can also consider a system of (6); we can replace the solution $u(x)$ and the component $a_{\alpha\beta}(x)$ by $u(x) \in \mathbb{R}^N$ and $(a^{ij}_{\alpha\beta}(x))_{i,j\in\{1,\cdots,N\}} \in \mathbb{R}^{N\times N}$, $N > 1$, respectively. Then the equation is

$$-\partial_\alpha\big(a_{\alpha\beta}(x)\partial_\beta u\big) \,=\, 0\,, \quad x \in B(0,1)\,. \tag{7}$$

Now we derive the Caccioppoli inequality. Let $0 < \rho < r \leq 1$ and let $\varphi \in C^\infty_{\mathrm{c}}(B(0,1))$ be a cut-off function such that

$$\operatorname{supp}\varphi \subset B(0,r)\,, \qquad \varphi(x) \equiv 1\,,\ x \in B(0,\rho)\,, \qquad \|\varphi\|_{L^\infty} \leq 2(r-\rho)^{-1}\,.$$

Then by testing $\varphi^2 u$ against the equation (6) we see that

$$\begin{aligned} 0 \,&=\, \int_{B(0,1)} \big(\mathbf{a}(x)\nabla u\big)\cdot\nabla(u\varphi^2)\\ &=\, \int_{B(0,1)} \big(\mathbf{a}(x)\nabla u\big)\cdot\nabla u\varphi^2 + \int_{B(0,1)} 2\big(\mathbf{a}(x)\nabla u\big)\cdot\nabla\varphi u\varphi\,. \end{aligned} \tag{8}$$

Since $\Lambda|\nabla u|^2 \leq \big(\mathbf{a}(x)\nabla u\big)\cdot\nabla u$, we have from (8) and the Hölder inequality,

$$\begin{aligned}\Lambda\int_{B(0,1)} |\varphi\nabla u|^2 &\leq \int_{B(0,1)} |2\big(\mathbf{a}(x)\nabla u\big)\cdot\nabla\varphi u\varphi|\\ &\leq 2\|\mathbf{a}\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\Big(\int_{B(0,1)} |\varphi\nabla u|^2\Big)^{\frac12}\Big(\int_{B(0,r)\setminus B(0,\rho)} |u|^2\Big)^{\frac12}\,.\end{aligned}$$

Then we have

$$\int_{B(0,r)} |\varphi \nabla u|^2 \leq \frac{C}{(r-\rho)^2} \int_{B(0,r)\setminus B(0,\rho)} |u|^2 \,, \tag{9}$$

where $C = C(\Lambda, L)$. The inequality (9) is called the Caccioppoli inequality. By the same computation, we can also prove a generalization of (9) for any $\xi \in \mathbb{R}$,

$$\int_{B(0,r)} |\varphi \nabla u|^2 \leq \frac{C}{(r-\rho)^2} \int_{B(0,r)\setminus B(0,\rho)} |u-\xi|^2 \,. \tag{10}$$

Here the constant $C = C(\Lambda, L)$ does not depend on $\xi \in \mathbb{R}$.

Widman's hole filling trick

We show an application of the Caccioppoli inequality. Let $r = 1$ and $\rho = \frac{1}{2}$ and set $\xi = \int_{B(0,1)\setminus B(0,\frac{1}{2})} u$. Then we have from (10) and the Poincaré inequality,

$$\begin{aligned} \int_{B(0,\frac{1}{2})} |\varphi \nabla u|^2 &\leq C \int_{B(0,1)\setminus B(0,\frac{1}{2})} |\nabla u|^2 \\ &= C\Big(\int_{B(0,1)} |\nabla u|^2 - \int_{B(0,\frac{1}{2})} |\nabla u|^2 \Big) \,, \end{aligned}$$

which implies

$$\int_{B(0,\frac{1}{2})} |\nabla u|^2 \leq \frac{C}{C+1} \int_{B(0,1)} |\nabla u|^2 \,.$$

By iterating this procedure $k \in \mathbb{N}$ times, we see that

$$\int_{B(0,\frac{1}{2^k})} |\nabla u|^2 \leq (\frac{C}{C+1})^k \int_{B(0,1)} |\nabla u|^2 \,.$$

Finally, by setting $\alpha = \frac{\log(\frac{C+1}{C})}{2\log 2}$, we can prove for any $\rho \in (0, \frac{1}{2})$,

$$\int_{B(0,\rho)} |\nabla u|^2 \leq C\rho^{2\alpha} \int_{B(0,1)} |\nabla u|^2 \,. \tag{11}$$

From the inequality (11), in the $\underline{d = 2}$ case, we can prove that $u \in C^{0,\alpha}(B(0, \frac{1}{2}))$ by the Morrey theorem. On the other hand, in the higher dimension $\underline{d \geq 3}$ case, we need the theory of De Giorgi, Nash, and Moser in order to prove $u \in C^{0,\alpha}$. We also mention counter-examples for system as in (7) for the $d \geq 3$ case in Giaquinta [7].

1.3 $C^{1,\alpha}$ improved regularity

We consider the following problem with $\epsilon \in (0, 1)$:

$$-\nabla \cdot \mathbf{a}(\frac{x}{\epsilon})\nabla u^\epsilon \,=\, 0 \,, \quad x \in B(0,1) \,, \tag{$*$}$$

where a matrix function $\mathbf{a} = \mathbf{a}(y) \in \mathbb{R}^{d\times d}$ belongs to the class $\mathcal{A}_{\text{per}}(\Lambda, L)$, $\Lambda \in (0,\infty)$, $L \in (0, \infty)$, which is defined as

$$\mathcal{A}_{\text{per}}(\Lambda, L) \,=\, \left\{ \mathbf{a} = \{a_{\alpha\beta}\}_{\alpha,\beta\in\{1,\ldots,d\}} \;\middle|\; \begin{array}{l} a_{\alpha\beta}(y) \text{ is measurable for } \alpha,\beta \in \{1,\ldots,d\}\,, \\ \mathbf{a} \text{ is elliptic with a constant } \Lambda\,, \\ \|\mathbf{a}\|_{L^\infty} \leq L\,, \quad \text{and } \mathbf{a}(y) \text{ is } \mathbb{Z}^d\text{-periodic} \end{array} \right\}.$$

Goal: Regularity estimates for the solutions to $(*)$ which is uniform in $\epsilon \in (0,1)$.

Homogenization

For the homogenization of the equation $(*)$ in the limit $\epsilon \to 0$, we need the following $\mathbb{Z}^d$-periodic (cell) corrector $\chi = \chi(y) \in \mathbb{R}$ satisfying

$$-\nabla \cdot \mathbf{a}(y)\nabla(y + \chi(y)) \,=\, 0\,, \quad y \in \mathbb{R}^d\,,$$
$$\fint_{\mathbb{T}^d} \chi \,=\, 0\,.$$

The function $y + \chi(y)$ is called $\mathbf{a}$-harmonic function. By using the corrector, we can make an ansatz for the solution $u^\epsilon = u^\epsilon(x)$ of $(*)$ as

$$u^\epsilon(x) \sim \bar{u}(x) + \epsilon\chi(\frac{x}{\epsilon}) \cdot \nabla\bar{u}(x)\,,$$

and if $x_0 \in \mathbb{R}^d$ is sufficiently close to $x \in \mathbb{R}^d$ then we also have

$$u^\epsilon(x) \sim \bar{u}(x_0) + \epsilon\big(\frac{x - x_0}{\epsilon} + \chi(\frac{x}{\epsilon})\big) \cdot \nabla\bar{u}(x_0)\,.$$

Here $\bar{u} = \bar{u}(x)$ is a solution to the homogenized equation of $(*)$

$$-\nabla \cdot \bar{\mathbf{a}}\nabla\bar{u} \,=\, 0\,, \quad x \in B(0,1)\,,$$

where the constant $\bar{\mathbf{a}} \in \mathbb{R}^{d\times d}$ is given by

$$\bar{\mathbf{a}} \,=\, \fint_{\mathbb{T}^d} (\bar{\mathbf{a}}(y) + \bar{\mathbf{a}}(y)\nabla\chi(y))\,.$$

Now we prove a key lemma concerning the convergence of the solutions to $(*)$.

Lemma 1.1 *Let $a \in \mathcal{A}_{\mathrm{per}}(\Lambda, L)$ and let a sequence $\{\epsilon_k\}$ satisfy $\epsilon_k \to 0$. Assume that a family of solutions $\{u_k\}$ of*

$$-\nabla \cdot \mathbf{a}(\frac{x}{\epsilon_k})\nabla u_k \,=\, 0\,, \quad x \in B(0,1) \tag{12}$$

is uniformly bounded in $W^{1,2}(B(0,1))$. Then, up to a subsequence of $\{u_k\}$, we have

$$\begin{aligned} u_k &\to \bar{u} \quad \text{in } L^2(B(0,1))\,,\\ \nabla u_k &\rightharpoonup \nabla\bar{u} \quad \text{in } L^2(B(0,1))^d\,,\\ a(\frac{x}{\epsilon_k})\nabla u_k &\rightharpoonup \bar{a}\nabla\bar{u} \quad \text{in } L^2(B(0,1))^d\,. \end{aligned} \tag{13}$$

Proof: We make a simplification by assuming that $\mathbf{a} = \{a_{\alpha\beta}\}_{\alpha,\beta\in\{1,\ldots,d\}}$ is a symmetric matrix, namely that $a_{\alpha\beta} = a_{\beta\alpha}$ for all $\alpha, \beta \in \{1, \ldots, d\}$. Since $\{\nabla u_k\}$ is uniformly bounded in $L^2(B(0,1))^d$, we know that there exists a function $\xi \in L^2(B(0,1))^d$ such that

$$\mathbf{a}(\frac{x}{\epsilon_k})\nabla u_k \rightharpoonup \xi \quad \text{in } L^2(B(0,1))^d\,. \tag{14}$$

Note that ξ satisfies $\nabla \cdot \xi = 0$. We apply the oscillating test function method by Murat and Tartar in late 70's. Let $\varphi \in C_c^\infty(B(0,1))$ and $\beta \in \{1, \ldots, d\}$. Then by testig the function

$$\varphi(x)(x_\beta + \epsilon_k \chi_\beta(\frac{x}{\epsilon_k})) \in \mathbb{R}$$

against the equation (12) and applying the integration by parts, we observe that

$$\begin{aligned}&\int_{B(0,1)} \big(\mathbf{a}(\frac{x}{\epsilon_k})\nabla u_k\big) \cdot \nabla\varphi(x_\beta + \epsilon_k \chi_\beta(\frac{x}{\epsilon_k}))\\ &= -\int_{B(0,1)} \big(\mathbf{a}(\frac{x}{\epsilon_k})\nabla u_k\big) \cdot \nabla(x_\beta + \epsilon_k \chi_\beta(\frac{x}{\epsilon_k}))\varphi\,.\end{aligned} \tag{15}$$

The first line in (15) converges to, in the limit $k \to \infty$,

$$\int_{B(0,1)} \xi \nabla\varphi x_\beta = -\int_{B(0,1)} \xi_\beta \varphi\,. \tag{16}$$

Here the integration by parts is applied combined with $\nabla \cdot \xi = 0$. On the other hand, the limit of the second line in (15) is computed as

$$\begin{aligned}&-\int_{B(0,1)} \nabla u_k \cdot \mathbf{a}(\frac{x}{\epsilon_k})\nabla(x_\beta + \epsilon_k \chi_\beta(\frac{x}{\epsilon_k}))\varphi\\ &= \int_{B(0,1)} u_k \mathbf{a}(\frac{x}{\epsilon_k})\nabla(x_\beta + \epsilon_k \chi_\beta(\frac{x}{\epsilon_k})) \cdot \nabla\varphi\\ &\to \int_{B(0,1)} \bar{u}\bar{a}_\beta \cdot \nabla\varphi = -\int_{B(0,1)} (\bar{a}\nabla\bar{u})_\beta \varphi\,.\end{aligned} \tag{17}$$

Thus we obtain $\xi_\beta = (\bar{a}\nabla\bar{u})_\beta$ for any $\beta \in \{1, \ldots, d\}$, and hence $\xi = (\bar{a}\nabla\bar{u})$. Then (14) leads to the last line of (13). The proof is complete. □

Next we state a uniform estimate to the problem $(*)$.

Theorem 1.2 (Avellaneda and Lin, uniform Lipschitz estimates) *For all $\epsilon \in (0, \infty)$, for all $a \in \mathcal{A}_{\mathrm{per}}(\Lambda, L)$ with $a \in C^{0,\mu}(\mathbb{R}^d)$ and $[a]_{C^{0,\mu}} \le M$, and for all solutions u^ϵ to $(*)$, there exists a constant $C = C(d, \Lambda, L, M) \in (0, \infty)$ such that we have*

$$\|\nabla u^\epsilon\|_{L^\infty(B(0,\frac{1}{2}))} \le C\|u^\epsilon\|_{L^2(B(0,1))}\,. \tag{18}$$

Sketch of the proof of Theorem 1.2: The proof consists of three steps.

Step (i): Improvement of flatness (corresponding to Lemma 1.4)
We apply the compactness argument and use the regularity for the limit equation of $(*)$.

Step (ii): Iteration of Step (i) (corresponding to Lemma 1.5)
We iterate the argument in Step (i) and go down to the scale ϵ.

Step (iii): Blow-up step
We apply classical regularity theory for the scale below ϵ. □

We prepare a lemma for a characterization of Hölder continuity.

Lemma 1.3 (Campanato) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then for any $\alpha \in (0,1]$ we have*

$$C^{0,\alpha}(\overline{\Omega}) \simeq \mathcal{L}^{2,\lambda}(\Omega)\,, \qquad \lambda \,=\, d+2\alpha \in (d, d+2]\,, \tag{19}$$

where the function space $\mathcal{L}^{2,\lambda}(\Omega)$ is defined as

$$\mathcal{L}^{2,\lambda}(\Omega) \,=\, \left\{ u \in L^2(\Omega) \mid [u]_{2,\lambda} \,=\, \sup_{\substack{x_0\in\Omega,\\ \rho\in(0,\infty)}} \rho^{-\lambda} \int_{\Omega\cap B(x,\rho)} |u-(u)_{x_0,\rho}|^2 < \infty \right\}. \tag{20}$$

Here we have set for $x_0 \in \mathbb{R}^d$ and $\rho \in (0,\infty)$,

$$(u)_{x_0,\rho} \,=\, \fint_{\Omega\cap B(x,\rho)} u\,.$$

The next lemma will be used in Step (i) of the proof of Theorem 1.2.

Lemma 1.4 *Let $\alpha \in (0,1)$. Then there exist constants $\theta \in (0,\frac{1}{2})$ and $\epsilon_0 \in (0,\infty)$ such that for all $a \in \mathcal{A}_{\mathrm{per}}(\Lambda, L)$, for all $\epsilon \in (0,\epsilon_0)$, and for all solutions u^ϵ to $(*)$, if*

$$\fint_{B(0,1)} |u^\epsilon|^2 \le 1$$

holds, then we have

$$\fint_{B(0,\theta)} |u^\epsilon(x) - (u^\epsilon)_{0,\theta} - (\nabla u^\epsilon)_{0,\theta}\cdot(x+\epsilon\chi(\frac{x}{\epsilon}))|^2 \le \theta^{2+2\alpha}\,.$$

Proof: Step (i): Choice of θ
The ϵ-zero limit equation of $(*)$ is given by

$$-\nabla\cdot\bar{\mathbf{a}}\nabla\bar{u} \,=\, 0\,, \quad x\in B(0,\frac{1}{2})\,,$$

where $\bar{\mathbf{a}}$ is an elliptic constant matrix with constant Λ and $|\bar{\mathbf{a}}| \le L$. Classical regularity theory implies $\bar{u} \in C^2(B(0,\frac{1}{4}))$. Then by the Campanato inequality we have

$$\fint_{B(0,\theta)} |\bar{u}(x) - (\bar{u})_{0,\theta} - (\nabla\bar{u})_{0,\theta}\cdot x|^2 \le C\theta^4\,, \tag{21}$$

where C is independent of θ. We choose $\theta \in (0,\frac{1}{2})$ sufficiently small so that

$$C\theta^4 < \theta^{2+2\alpha}\,. \tag{22}$$

Step (ii): Contradiction
Assume that there exist sequences $\{\epsilon_k\}$ and $\{u_k\}$ such that $\epsilon_k \to 0$ and $u_k = u_k(x)$ satisfies

$$-\nabla\cdot a(\frac{x}{\epsilon_k})\nabla u_k \,=\, 0\,, \quad x\in B(0,1)\,,$$
$$\fint_{B(0,1)} |u_k|^2 \le 1\,,$$

and

$$\fint_{B(0,\theta)} |u_k(x) - (u_k)_{0,\theta} - (\nabla u_k)_{0,\theta} \cdot (x + \epsilon_k \chi(\frac{x}{\epsilon_k}))|^2 > \theta^{2+2\alpha}\,. \tag{23}$$

Then by the Caccioppoli inequality $\{u_k\}$ is uniformly bounded in $W^{1,2}(B(0,\frac{1}{2}))$. In a similar manner as in the proof of Lemma 1.1 we can prove that

$$u_k \to \bar{u} \quad \text{in } L^2(B(0,\frac{1}{2}))\,, \tag{24}$$

$$\nabla u_k \rightharpoonup \nabla \bar{u} \quad \text{in } L^2(B(0,\frac{1}{2}))\,, \tag{25}$$

and

$$-\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = 0\,, \quad x \in B(0,\frac{1}{2})\,.$$

Here $\bar{\mathbf{a}}$ is an elliptic constant matrix with constant Λ and $|\bar{\mathbf{a}}| \le L$. By (24)–(25) we see that

$$(u_k)_{0,\theta} \to (\bar{u})_{0,\theta}\,, \qquad (\nabla u_k)_{0,\theta} \to (\nabla \bar{u})_{0,\theta}\,. \tag{26}$$

Since the corrector χ is bounded in $\mathbb{R}^d$, it is easy to see that

$$\fint_{B(0,\theta)} |\epsilon_k \chi(\frac{x}{\epsilon_k}))|^2 \le C\epsilon_k^2\,.$$

Thus by taking the limit $k \to \infty$ of (23) we obtain

$$\begin{aligned} \theta^{2+2\alpha} &\le \limsup_{k\to\infty} \fint_{B(0,\theta)} |u_k(x) - (u_k)_{0,\theta} - (\nabla u_k)_{0,\theta} \cdot x|^2 \\ &\le \fint_{B(0,\theta)} |\bar{u}(x) - (\bar{u})_{0,\theta} - (\nabla \bar{u})_{0,\theta} \cdot x|^2 \\ &< \theta^{2+2\alpha} \end{aligned}$$

from (21)–(22) in Step (i). Hence we have a contradiction. This completes the proof. □

The next lemma corresponds to Step (ii) in the proof of Theorem 1.2.

Lemma 1.5 *Let α, θ, and ϵ_0 be given in Lemma 1.4. Then for all $k \in \mathbb{N}$, for all $a \in \mathcal{A}_{\text{per}}(\Lambda, L)$, for all $\epsilon \in (0, \theta^{k-1}\epsilon_0)$, and for all solutions u^ϵ to $(*)$, if*

$$\fint_{B(0,1)} |u^\epsilon|^2 \le 1 \tag{27}$$

holds, then we have

$$\begin{aligned} &\inf_{\substack{a\in\mathbb{R},\\ b\in\mathbb{R}^d}} \fint_{B(0,\theta^k)} |u^\epsilon(x) - a - b \cdot (x + \epsilon\chi(\frac{x}{\epsilon})|^2 \\ &\quad \le \fint_{B(0,\theta^k)} |u^\epsilon(x) - a_k^\epsilon - b_k^\epsilon \cdot (x + \epsilon\chi(\frac{x}{\epsilon})|^2 \\ &\quad \le \theta^{(2\alpha+2)k}\,, \end{aligned} \tag{28}$$

where the constants $a_k^\epsilon \in \mathbb{R}$ and $b_k^\epsilon \in \mathbb{R}^d$ respectively satisfy

$$|a_k^\epsilon| \le \theta^{-\frac{\theta}{2}}(1+\theta^{2\alpha+2}+\cdots+\theta^{(2\alpha+2)(k-1)})\,, \tag{29}$$

$$|b_k^\epsilon| \le C\theta^{-\frac{\theta}{2}}(1+\theta^{2\alpha+1}+\cdots+\theta^{(2\alpha+1)(k-1)})\,. \tag{30}$$

Remark 1.6 The iteration would be easier if there is no correction χ; we would have

$$a_k^\epsilon \;=\; (u^\epsilon)_{0,\theta^k}\,, \qquad b_k^\epsilon \;=\; (\nabla u^\epsilon)_{0,\theta^k}\,.$$

Proof: The proof is by iteration on $k \in \mathbb{N}$.
$\underline{k=1}$: The estimate (28) follows from Lemma 1.4. By the assumption (27) and the equation $(*)$ we have

$$|(u^\epsilon)_{0,\theta}| \le \theta^{-\frac{d}{2}}\,, \qquad |(\nabla u^\epsilon)_{0,\theta})| \le C\theta^{-\frac{d}{2}}\,.$$

$\underline{k>1}$: Assume that the assertions in the lemma hold for all $k-1\ge 1$. Then we set

$$U^\epsilon(x) \;=\; \frac{u^\epsilon(\theta^{k-1}x) - a_{k-1}^\epsilon - b_{k-1}^\epsilon(\theta^{k-1}x+\epsilon\chi(\frac{\theta^{k-1}x}{\epsilon}))}{\theta^{(2\alpha+2)(k-1)}}\,.$$

By the iteration assumption we have

$$\begin{aligned}\fint_{B(0,1)} |U^\epsilon|^2 &\le \frac{1}{\theta^{(2\alpha+2)(k-1)}}\fint_{B(0,\theta^{k-1})} |u^\epsilon(x) - a_{k-1}^\epsilon - b_{k-1}^\epsilon(x+\epsilon\chi(\frac{x}{\epsilon}))|^2 \\ &\le 1\,,\end{aligned} \tag{31}$$

and

$$-\nabla\cdot a(\frac{\theta^{k-1}x}{\epsilon})\nabla U^\epsilon \;=\; 0\,, \quad x\in B(0,1)\,. \tag{32}$$

Then Lemma 1.4 and the assumption $\epsilon \in (0,\theta^{k-1}\epsilon_0)$ lead to

$$\begin{aligned}\theta^{2\alpha+2} &\ge \fint_{B(0,\theta)} |U^\epsilon(x) - (U^\epsilon)_{0,\theta} - (\nabla U^\epsilon)_{0,\theta}\cdot(x+\frac{\epsilon}{\theta^{k-1}}\chi(\frac{\theta^{k-1}x}{\epsilon}))|^2 \\ &= \frac{1}{\theta^{(2\alpha+2)(k-1)}}\fint_{B(0,\theta^k)} |u^\epsilon(x) - a_k^\epsilon - b_k^\epsilon(x+\epsilon\chi(\frac{x}{\epsilon}))|^2\,,\end{aligned}$$

where we set

$$a_k^\epsilon \;=\; a_{k-1}^\epsilon + \theta^{(2\alpha+2)(k-1)}(U^\epsilon)_{0,\theta}\,, \qquad b_k^\epsilon \;=\; b_{k-1}^\epsilon + \theta^{(2\alpha+1)(k-1)}(\nabla U^\epsilon)_{0,\theta}\,.$$

Thus we have (28). We also have from (31) and (32),

$$|(U^\epsilon)_{0,\theta}| \le \theta^{-d}\,, \qquad |(\nabla U^\epsilon)_{0,\theta}| \le C\theta^{-d}\,.$$

This completes the proof. □

From the iteration argument in the proof of Lemma 1.5, we find that for any $\rho \in (\frac{\epsilon}{\epsilon_0},\frac{1}{2})$,

$$\inf_{\substack{a\in\mathbb{R},\\ b\in\mathbb{R}^d}} \fint_{B(0,\rho)} |u^\epsilon(x) - a - b\cdot(x+\epsilon\chi(\frac{x}{\epsilon})|^2 \le \rho^{2\alpha+2}\fint_{B(0,1)} |u^\epsilon|^2\,. \tag{33}$$

This inequality is a key estimate in the next subsection.

1.4 Liouville theorems

We consider

$$-\nabla \cdot a(y)\nabla u \;=\; 0\,, \quad y \in \mathbb{R}^d\,,$$

where $a \in \mathcal{A}_{\mathrm{per}}(\Lambda, L)$. The theorem is the following:

Theorem 1.7 (i) *If there exist constants $C \in (0,\infty)$ and $\sigma \in (0,1)$ such that*

$$\fint_{B(0,R)} |u|^2 \le CR^{2\sigma}$$

holds for all $R \geq 2017$, then there exists a number $a \in \mathbb{R}$ such that

$$u(y) \;=\; a\,, \quad y \in \mathbb{R}^d\,.$$

(ii) *If there exist constants $C \in (0,\infty)$ and $\sigma \in (0,1)$ such that*

$$\fint_{B(0,R)} |u|^2 \le CR^{2\sigma+2}$$

holds for all $R \geq 2017$, then there exist numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ such that

$$u(y) \;=\; a + b(y + \chi(y))\,, \quad y \in \mathbb{R}^d\,.$$

Proof: We only prove the claim (ii). We note that $u = u(y)$ satisfies

$$-\nabla \cdot a(y)\nabla u \;=\; 0\,, \quad y \in B(0,R)\,.$$

We fix $\alpha \in (0,\sigma)$. By rescaling the inequality (33) we have for any fixed $r \in [1, \frac{R}{2})$,

$$\inf_{\substack{a\in\mathbb{R},\\ b\in\mathbb{R}^d}} \fint_{B(0,r)} |u(y) - a - b\cdot(y + \epsilon\chi(y)|^2 \le \big(\frac{r}{R}\big)^{2+2\alpha} \fint_{B(0,R)} |u|^2\,.$$

Then from the choice of α we see that

$$\big(\frac{r}{R}\big)^{2+2\alpha} \fint_{B(0,R)} |u|^2 \le r^{2\alpha+2} R^{2(\alpha-\sigma)} \to 0$$

in the limit $R \to \infty$. Hence we obtain the claim (ii). This completes the proof. $\square$

2 Lectures 3 – 4: Partial regularity for Navier-Stokes

2.1 Introduction

In this lecture we consider the three-dimensional Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p \;=\; 0\,, \\ \qquad\qquad\qquad\quad \nabla \cdot u \;=\; 0\,. \end{cases} \tag{NSE}$$

Here $u = u(x,t) \in \mathbb{R}^3$ and $p = p(x,t) \in \mathbb{R}$ respectively denote the velocity field and the pressure field of the fluid. Our aim in this lecture is the following claim.

Aim Let (u,p) be a "sufficiently nice" solution to (NSE). Then if

$$\int_{B(0,1)\times(-1,0)} |u|^3 + |p|^{\frac{3}{2}} \le \epsilon_* \tag{34}$$

holds with some small positive constant $\epsilon_* \in (0,\infty)$, then the solution $u = u(x,t)$ is regular in $B(0,\frac{1}{2}) \times (-\frac{1}{4},0)$. This is one of the Caffarelli-Kohn-Nirenberg (ϵ-regular) criteria.

For fixed $(x_0,t_0) \in \mathbb{R}^3 \times (-\infty,0]$, we denote by $Q_r(x_0,t_0)$ the parabolic cylinder centered at (x_0,t_0) with radius $r \in (0,\infty)$:

$$Q_r(x_0,t_0) \,=\, B(x_0,r) \times (-r^2 + t_0, t_0)\,.$$

For the case $(x_0,t_0) = (0,0)$, we denote $Q_r(x_0,t_0)$ by Q_r for simplicity.

2.2 Fundamental facts

To start with, let us mention a few fundamental facts about the Navier-Stokes equations.

Weak solution
The pair (u,p) is a weak solution to (NSE) if we have

$$-\langle u, \partial_t \varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle + \langle \nabla u, \nabla\varphi\rangle - \langle p, \nabla\cdot\varphi\rangle \,=\, 0\,, \quad \varphi \in C_{\rm c}^\infty(\mathbb{R}^3\times\mathbb{R})^3\,, \tag{35}$$

and

$$\langle u, \nabla\phi\rangle \,=\, 0\,, \quad \phi \in C_{\rm c}^\infty(\mathbb{R}^3\times\mathbb{R})\,. \tag{36}$$

Evolution problem vs. regularity problem

Evolution problem: The equations are considered in $\Omega\times(0,T)$, where Ω is a domain in $\mathbb{R}^3$ and $T \in (0,\infty)$, together with the boundary condition at $\partial\Omega$ and initial condition at $t = 0$.

Regularity problem: The equations are considered locally in a space-time domain Q, without imposing any initial condition nor boundary condition.

Local energy equality
Assume that a solution (u,p) to (NSE) is smooth. Then for all $\varphi \in C_{\rm c}^\infty(\mathbb{R}^3\times\mathbb{R})^3$ and for all $-\infty < t' < t < \infty$, we have the following local energy equality

$$\begin{aligned}&\int_{\mathbb{R}^3} |u(\cdot,t)|^2\varphi(\cdot,t) + 2\int_{t'}^t\int_{\mathbb{R}^3} |\nabla u|^2\varphi \\ &= \int_{\mathbb{R}^3} |u(\cdot,t')|^2\varphi(\cdot,t') + \int_{t'}^t\int_{\mathbb{R}^3} |u|^2(\partial_t\varphi + \Delta\varphi) + (|u|^2 + 2p)u\cdot\nabla\varphi\,.\end{aligned} \tag{LEE}$$

Pressure
Formally operating $\nabla\cdot$ to the first equation of (NSE) we have

$$-\Delta p \,=\, \nabla\cdot(u\cdot\nabla u)\,.$$

Hence the regularity in space is not influenced by the nonlocal effects of the pressure p, and however, the regularity in time is influenced. This is indeed the case for the next example by Serrin: let $a(t) \in \mathbb{R}$ be any bounded function and $\Phi(x) \in \mathbb{R}$ be any harmonic function. Then the pair

$$u(x,t) \;=\; a(t)\nabla\Phi(x)\,, \qquad p(x,t) \;=\; -a'(t)\nabla\Phi(x) - \frac{1}{2}|u|^2\,,$$

gives a weak solution to (NSE). The regularity in time of $\partial_t u$ is same as the one of p.

Scaling
Let $\lambda \in (0,\infty)$. If $u = u(x,t)$ is a solution to (NSE), then we see that

$$u_\lambda(x,t) \;=\; \lambda u(\lambda x, \lambda^2 t)$$

is also a solution to (NSE). In view of scale-invariance, regularity results read "If some scale invariant quantity $F(u,p,r)$ is small, then the solution is regular". In the 2d case, the energy is scale invariant. In the 3d case, for the initial value problem with $u_0 \in L^2_\sigma$ we have

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(\cdot,t)|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \le \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2\,.$$

We note that the energy is supercritical for the $3d$ case.

Local Suitable Weak Solution
As a "nice class of solutions", we introduce the following local suitable weak solutions.

Definition 2.1 *A pair (u,p) in Q_1 is a Local Suitable Weak Solution (LSWS) of* (NSE) *if*
(1) (u,p) *is a weak solution of* (NSE).
(2) (u,p) *satisfies* $u \in L^\infty_t L^2_x(Q_1) \cap L^2_t H^1_x(Q_1)$ *and* $p \in L^{\frac{3}{2}}_{t,x}(Q_1)$.
(3) (u,p) *satisfies the local energy inequality for all* $\varphi \in C^\infty_{\rm c}(B(0,1)\times(-1,0])$, $\varphi \ge 0$;

$$\begin{aligned}&\int_{\mathbb{R}^3} |u(\cdot,t)|^2\varphi(\cdot,t) + 2\int_{-1}^t \int_{\mathbb{R}^3} |\nabla u|^2\varphi \\ &\le \int_{-1}^t \int_{\mathbb{R}^3} |u|^2(\partial_t\varphi + \Delta\varphi) + (|u|^2 + 2p)u\cdot\nabla\varphi\,, \quad \text{a.e. } t \in (-1,0)\,.\end{aligned} \tag{LEI}$$

2.3 $u + p$ criteria

We prove the following theorem.

Theorem 2.2 *There exist constants $\epsilon_* \in (0,\infty)$, $C \in (0,\infty)$, and $\alpha \in (0,1)$ such that for all LSWS (u,p) to* (NSE) *in Q_1, if*

$$\int_{Q_1} |u|^3 + |p|^{\frac{3}{2}} \le \epsilon_* \tag{S-ϵ_*}$$

holds, then we have $u \in C^\alpha_{\rm par}(\overline{Q_{\frac{1}{2}}})$ and

$$[u]_{C^\alpha_{\rm par}(\overline{Q_{\frac{1}{2}}})} \le C\,.$$

Here $u \in C^\alpha_{\rm par}(\overline{Q_{\frac{1}{2}}})$ means that $u \in L^\infty(\overline{Q_{\frac{1}{2}}})$ and $u = u(x,t)$ satisfies

$$|u(x,t) - u(\hat{x},\hat{t})| \le [u]_{C^\alpha_{\rm par}(\overline{Q_{\frac{1}{2}}})}(|x-\hat{x}|^{2\alpha} + |t-\hat{t}|^\alpha)\,, \quad (x,t)\,,(\hat{x},\hat{t}) \in \overline{Q_{\frac{1}{2}}}\,.$$

Firstly we mention the Campanato characterization of Hölder continuity: let $p \in [1, \infty)$. Then $u \in C^{\alpha}_{\mathrm{par}}(\overline{Q_1})$ if and only if $u \in L^p(\overline{Q_1})$ and

$$\sup_{\substack{r\in(0,\infty),\\ (x_0,t_0)\in Q_1}} \frac{1}{r^{\alpha p}} \fint_{Q_r(x_0,t_0)} |u - (u)_r|^p < \infty\,.$$

Here the integral $\fint_{Q_r}$ on the parabolic cylinder Q_r is defined as for $r \in (0, \infty)$,

$$\fint_{Q_r(x_0,t_0)} f \;=\; \frac{1}{|Q_r|}\int_{Q_r(x_0,t_0)} f(x,t)\,, \qquad |Q_r| \sim r^5\,,$$

and $(u)_r$ is defined as for $r \in (0, \infty)$,

$$(u)_r \;=\; \fint_{Q_r} u\,.$$

Let us define the quantity $F(u, p, r)$ by for $r \in (0, \infty)$,

$$F(u,p,r) \;=\; \frac{1}{r^2}\int_{Q_r} |u|^3 + |p|^{\frac{3}{2}}\,.$$

Then the quantity $F(u, p, r)$ is invariant under the scaling of the Navier-Stokes equations:

$$F(u,p,r) \;=\; F(u_\lambda, p_\lambda, \frac{r}{\lambda})\,, \quad \lambda > 0\,.$$

Moreover, we have the rescaled version of Theorem 2.2 as follows: there exist constants $\epsilon_* \in (0, \infty)$, $C \in (0, \infty)$, and $\alpha \in (0, 1)$ such that if there exists $r \in (0, \infty)$ such that if

$$F(u,p,r) \le \epsilon_* \tag{37}$$

holds, then we have

$$[u]_{C^{\alpha}_{\mathrm{par}}(\overline{Q_{\frac{r}{2}}})} \le C\,.$$

Sketch of the proof of Theorem 2.2: The proof is due to Lin [12]. We set

$$\mathrm{osc}(u,p,r) \;=\; \left(\fint_{Q_r} |u-(u)_r|^3\right)^{\frac{1}{3}} \,+\, r\left(\fint_{Q_r} |p-(p)_r(t)|^{\frac{3}{2}}\right)^{\frac{2}{3}}\,,$$

where the function $(p)_r = (p)_r(t)$ is defined as

$$(p)_r(t) \;=\; \fint_{B(0,r)} p(\cdot,t)\,.$$

Step (i): Improvement of flatness (corresponding to Lemma 2.3)
We prove the following claim: if (u, p) satisfies (S-ϵ_*), then there exist some $\theta \in (0, \frac{1}{2})$ and $\alpha \in (0, \frac{1}{3})$ such that we have

$$\mathrm{osc}(u,p,\theta) \le \theta^{2\alpha}\epsilon_*\,.$$

Step (ii): Iteration of Step (i) (corresponding to Lemma 2.4)
In this step we consider the Navier-Stokes equations with drift $b \in \mathbb{R}^3$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + b \cdot \nabla u - \Delta u + \nabla p \,=\, 0\,, \\ \nabla \cdot u \,=\, 0\,. \end{cases} \tag{NSEdrift}$$

and extend the notion of LSWS to (NSEdrift). □

The next lemma is used in Step (i) of the proof of Theorem 2.2.

Lemma 2.3 *There exist constants $\epsilon_0 \in (0, \infty)$, $\theta \in (0, \frac{1}{2})$, and $\alpha \in (0, \frac{1}{3})$ such that for all $b \in \mathbb{R}^3$ and for all LSWS (u, p) to* (NSEdrift) *in Q_1, if* (i) *the smallness condition*

$$\left(\fint_{Q_1} |u|^3 \right)^{\frac{1}{3}} + \left(\fint_{Q_1} |p|^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq \epsilon_0$$

holds and (ii) *b satisfies $|b| \leq 1$, then we have*

$$\operatorname{osc}(u, p, \theta) \leq \theta^{2\alpha} \epsilon_0\,.$$

Proof: Step (i): Limit equation
By fixing θ and α, we consider the limit (linear) equations

$$\begin{cases} \partial_t v + b \cdot \nabla v - \Delta v + \nabla q \,=\, 0\,, & (x,t) \in Q_{\frac{2}{3}}\,, \\ \nabla \cdot v \,=\, 0\,, & (x,t) \in Q_{\frac{2}{3}}\,, \end{cases} \tag{38}$$

with $|b| \leq 1$ and

$$\|v\|_{L^3(Q_{\frac{2}{3}})} \leq |Q_1|^{\frac{1}{3}}\,, \qquad \|q\|_{L^{\frac{3}{2}}(Q_{\frac{2}{3}})} \leq |Q_1|^{\frac{1}{3}}\,.$$

By the regularity theory for the Stokes (linear) problem, for the velocity v we have

$$v \in C^{\frac{1}{3}}_{\text{par}}(Q_{\frac{1}{3}})\,.$$

Hence, by Campanato's characterization of Hölder continuity, we see that for all $\theta \in (0, \frac{1}{3})$,

$$\left(\fint_{Q_\theta} |v - (v)_\theta|^3 \right)^{\frac{1}{3}} \leq C_0 \theta^{\frac{1}{3}}\,.$$

Next we consider the estimate for the pressure q. Since we have

$$-\Delta q \,=\, 0\,, \quad (x,t) \in Q_{\frac{2}{3}}\,,$$

from the regularity in space for harmonic equations, we see that for all $\theta \in (0, \frac{2}{3})$,

$$\left(\fint_{B(0,\theta)} |q(\cdot,t) - (q)_\theta(t)|^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq C\theta \left(\fint_{B(0,\frac{2}{3})} |q(\cdot,t)|^{\frac{3}{2}} \right)^{\frac{2}{3}}\,. \tag{39}$$

Then, by the integration in time and $\|q\|_{L^{\frac{3}{2}}(Q_{\frac{2}{3}})} \le |Q_1|^{\frac{1}{3}}$, we obtain for all $\theta \in (0, \frac{2}{3})$,

$$\theta\left(\fint_{Q_\theta} |q - (q)_\theta(t)|^{\frac{3}{2}}\right)^{\frac{2}{3}} \le C\theta^{\frac{2}{3}}\left(\fint_{Q_{\frac{2}{3}}} |q|^{\frac{3}{2}}\right)^{\frac{2}{3}} \le C_1\theta^{\frac{2}{3}}\,. \tag{40}$$

Finally we fix $\alpha \in (0, \frac{1}{3})$ and choose $\theta \in (0, \frac{1}{2})$ sufficiently small so that

$$C_0\theta^{\frac{1}{3}} + C_1\theta^{\frac{2}{3}} \le \frac{1}{2}\theta^{2\alpha}\,. \tag{41}$$

Step (ii): Beginning of contradiction argument
Assume that there exist sequences $\{\epsilon_k\}$ and $\{(u_k, p_k)\}$ such that $\epsilon_k \to 0$ and (u_k, p_k) is anx LSWS to (NSEdrift) in Q_1 satisfying

$$\left(\fint_{Q_1} |u_k|^3\right)^{\frac{1}{3}} + \left(\fint_{Q_1} |p_k|^{\frac{3}{2}}\right)^{\frac{2}{3}} = \epsilon_k$$

and

$$\left(\fint_{Q_\theta} |u_k - (u_k)_\theta|^3\right)^{\frac{1}{3}} + \theta\left(\fint_{Q_\theta} |p_k - (p_k)_\theta(t)|^{\frac{3}{2}}\right)^{\frac{2}{3}} > \theta^{2\alpha}\epsilon_k\,.$$

Firstly we rescale (u_k, p_k) by setting

$$v_k = \frac{u_k}{\epsilon_k}\,, \qquad q_k = \frac{p_k}{\epsilon_k}\,,$$

which leads to

$$\left(\fint_{Q_1} |v_k|^3\right)^{\frac{1}{3}} + \left(\fint_{Q_1} |q_k|^{\frac{3}{2}}\right)^{\frac{2}{3}} = 1\,, \tag{42}$$

$$\left(\fint_{Q_\theta} |v_k - (v_k)_\theta|^3\right)^{\frac{1}{3}} + \theta\left(\fint_{Q_\theta} |q_k - (q_k)_\theta(t)|^{\frac{3}{2}}\right)^{\frac{2}{3}} > \theta^{2\alpha}\,. \tag{43}$$

We see that (v_k, q_k) is an LSWS to

$$\begin{cases} \partial_t v_k + \epsilon_k v_k \cdot \nabla v_k + b \cdot \nabla v_k - \Delta v_k + \nabla q_k = 0\,, & (x,t) \in Q_1\,, \\ \nabla \cdot v_k = 0\,, & (x,t) \in Q_1\,. \end{cases} \tag{NSEdrift-ϵ_k}$$

From (42) we have weak convergences $v_k \rightharpoonup v$ in $L^3(Q_1)$ and $q_k \rightharpoonup q$ in $L^{\frac{3}{2}}(Q_1)$ and

$$\|v\|_{L^3(Q_1)} \le \liminf_{k\to\infty} \|v_k\|_{L^3(Q_1)} \le 1\,,$$
$$\|q\|_{L^{\frac{3}{2}}(Q_1)} \le \liminf_{k\to\infty} \|q_k\|_{L^{\frac{3}{2}}(Q_1)} \le 1\,.$$

Step (iii): Strong compactness
We will prove the following claim that up to a subsequence we have

$$v_k \to v \quad \text{in} \;\; L^3(Q_{\frac{2}{3}})\,.$$

From the local energy inequality of (NSEdrift-ϵ_k), we have for all $\varphi \in C_c^\infty(B(0,1) \times (-1,0])$, $\varphi \geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |v_k(\cdot,t)|^2 \varphi(\cdot,t) + 2\int_{-1}^t \int_{\mathbb{R}^3} |\nabla v_k|^2 \varphi \\
&\leq \int_{-1}^t \int_{\mathbb{R}^3} |v_k|^2(\partial_t \varphi + \Delta\varphi) + |v_k|^2 b \cdot \nabla\varphi + (\epsilon_k |v_k|^2 + 2q_k) v_k \cdot \nabla\varphi \\
&\leq C\,, \quad \text{a.e. } t \in (-1,0)\,,
\end{aligned} \tag{44}$$

which implies

$$\{v_k\} \text{ is uniformly bounded in } L_t^\infty L_x^2(Q_{\frac{2}{3}}) \cap L_t^2 H_x^1(Q_{\frac{2}{3}})\,. \tag{45}$$

Thus, in particular, we have by the Hölder inequality,

$$\{v_k\} \text{ is uniformly bounded in } L_{t,x}^{\frac{10}{3}}(Q_{\frac{2}{3}})\,. \tag{46}$$

Moreover, by using the equations (NSEdrift-ϵ_k) we have for all $\varphi \in C_c^\infty(Q_{\frac{2}{3}};\mathbb{R}^3)$,

$$\begin{aligned}
|\langle \partial_t v_k, \varphi\rangle| &\leq \left| \int_{-(\frac{2}{3})^2}^0 \int_{B(0,\frac{2}{3})} \nabla v_k \cdot \nabla\varphi \right| + \left| \int_{-(\frac{2}{3})^2}^0 \int_{B(0,\frac{2}{3})} ((\epsilon_k v_k + b)\cdot \nabla v_k)\cdot\varphi \right| \\
&\quad + \left| \int_{-(\frac{2}{3})^2}^0 \int_{B(0,\frac{2}{3})} q_k \nabla\cdot\varphi \right| \\
&\leq \|\nabla v_k\|_{L_t^{\frac{3}{2}} L_x^2(Q_{\frac{2}{3}})} \|\nabla\varphi\|_{L_t^3 L_x^2(Q_{\frac{2}{3}})} \\
&\quad + \|\epsilon_k v_k + b\|_{L_t^\infty L_x^2(Q_{\frac{2}{3}})} \|\nabla v_k\|_{L_t^{\frac{3}{2}} L_x^2(Q_{\frac{2}{3}})} \|\varphi\|_{L_t^3 L_x^\infty(Q_{\frac{2}{3}})} \\
&\quad + \|q_k\|_{L_{t,x}^{\frac{3}{2}}(Q_{\frac{2}{3}})} \|\nabla\varphi\|_{L_{t,x}^3(Q_{\frac{2}{3}})} \\
&\leq C\|\varphi\|_{L_t^3 H_x^2(Q_{\frac{2}{3}})}\,.
\end{aligned}$$

Thus we observe that

$$\partial_t v_k \in L_t^{\frac{3}{2}} (H_0^2(Q_{\frac{2}{3}}))_x'\,.$$

Hence by the Aubin-Lions-Rellich lemma we see that $\{v_k\}$ is precompact in $L_{t,x}^{\frac{3}{2}}(Q_{\frac{2}{3}})$. Then by the uniform bound in $L_{t,x}^{\frac{10}{3}}(Q_{\frac{2}{3}})$ in (46), we have for all $q \in [1, \frac{10}{3})$,

$$v_k \to v \quad \text{in } L^q(Q_{\frac{2}{3}})\,. \tag{47}$$

Step (iv): Passing to the limit

We take the limit $k \to \infty$ of (43). By observing that $v = \lim_{k\to\infty} v_k$ is a solution to the linear equations (38), and by using the bounds in (45)–(46) and the convergence (47), we obtain

$$\begin{aligned}
\theta^{2\alpha} &\leq \left(\fint_{Q_\theta} |v - (v)_\theta|^3 \right)^{\frac{1}{3}} + \limsup_{k\to\infty} \theta \left(\fint_{Q_\theta} |q_k - (q_k)_\theta(t)|^{\frac{3}{2}} \right)^{\frac{2}{3}} \\
&\leq C_0 \theta^{\frac{1}{3}} + \limsup_{k\to\infty} \theta \left(\fint_{Q_\theta} |q_k - (q_k)_\theta(t)|^{\frac{3}{2}} \right)^{\frac{2}{3}}.
\end{aligned} \tag{48}$$

We consider a control of the pressure. We decompose q_k as $q_k = h_k + g_k$. Here h_k satisfies

$$\begin{cases} \quad -\Delta h_k \, = \, 0 \, , \quad x \in B(0, \dfrac{2}{3}) \, , \\ (h_k)_\theta(t) \, = \, (q_k)_\theta(t) \, , \end{cases}$$

while g_k satisfies

$$\begin{cases} -\Delta g_k \, = \, \epsilon_k \nabla \cdot (v_k \cdot \nabla v_k) \, , \quad x \in B(0, \dfrac{2}{3}) \, , \\ \qquad g_k \, = \, 0 \, , \quad x \in \partial B(0, \dfrac{2}{3}) \, . \end{cases}$$

The Calderon-Zygmund estimates for g_k lead to

$$\|\nabla g_k(t)\|_{L_x^{\frac{9}{8}}(B(0,\frac{2}{3}))} \le \epsilon_k \|v_k(t) \cdot \nabla v_k(t)\|_{L_x^{\frac{9}{8}}(B(0,\frac{2}{3}))} \, .$$

By integrating in time and combining with the Poincaré-Sobolev inequality we have

$$\begin{aligned} \Big(\fint_{Q_{\frac{2}{3}}} |g_k|^{\frac{3}{2}} \Big)^{\frac{2}{3}} &\le C \|g_k\|_{L_t^{\frac{3}{2}}\left(-(\frac{2}{3})^2, 0 \, ; \, L_x^{\frac{9}{5}}(B(0,\frac{2}{3})\right)} \\ &\le C \|\nabla g_k\|_{L_t^{\frac{3}{2}}\left(-(\frac{2}{3})^2, 0 \, ; \, L_x^{\frac{9}{8}}(B(0,\frac{2}{3})\right)} \\ &\le C\epsilon_k \|v_k \cdot \nabla v_k\|_{L_t^{\frac{3}{2}}\left(-(\frac{2}{3})^2, 0 \, ; \, L_x^{\frac{9}{8}}(B(0,\frac{2}{3})\right)} \\ &\le C \epsilon_k \, , \end{aligned}$$

where the energy inequality (44) is applied to derive the last line. By a similar argument as we have derived (39) and (40) and $h_k = q_k - g_k$ we have

$$\begin{aligned} \theta \Big(\fint_{Q_\theta} |h_k - (h_k)_\theta(t)|^{\frac{3}{2}} \Big)^{\frac{2}{3}} &\le C\theta^{\frac{2}{3}} \bigg(\Big(\fint_{Q(0,\frac{2}{3})} |q_k|^{\frac{3}{2}} \Big)^{\frac{2}{3}} + \Big(\fint_{Q(0,\frac{2}{3})} |g_k|^{\frac{3}{2}} \Big)^{\frac{2}{3}} \bigg) \\ &\le C_1 \theta^{\frac{2}{3}} + C\epsilon_k \, . \end{aligned}$$

Then we see that

$$\begin{aligned} &\limsup_{k\to\infty} \theta \Big(\fint_{Q_\theta} |q_k - (q_k)_\theta(t)|^{\frac{3}{2}} \Big)^{\frac{2}{3}} \\ &\le \limsup_{k\to\infty} \theta \Big(\fint_{Q_\theta} |h_k - (h_k)_\theta(t)|^{\frac{3}{2}} \Big)^{\frac{2}{3}} + \limsup_{k\to\infty} \theta \Big(\fint_{Q_\theta} |g_k|^{\frac{3}{2}} \Big)^{\frac{2}{3}} \\ &\le C_1 \theta^{\frac{2}{3}} \, . \end{aligned}$$

Thus from (48) we find

$$\theta^{2\alpha} \le C_0 \theta^{\frac{1}{3}} + C_1 \theta^{\frac{2}{3}} \, .$$

On the other hand, from (41) in Step (i) we have $C_0\theta^{\frac{1}{3}} + C_1\theta^{\frac{2}{3}} < \frac{1}{2}\theta^{2\alpha}$. Hence we have a contradiction. This completes the proof of Lemma 2.3. □

The lemma in the next corresponds to Step (ii) in the proof of Theorem 2.2.

Lemma 2.4 *Let θ, α, and ϵ_0 be given in Lemma 2.3. Choose ϵ_0 sufficiently small so that*

$$\epsilon_0 \in (0, \frac{\theta^5}{2}]$$

if needed. Then for all $k \in \mathbb{N}$ and for all LSWS (u,p) to (NSEdrift) *in Q_{θ^k}, if* (i) *the smallness condition*

$$\operatorname{osc}(u,p,\theta^k) \leq \theta^{2\alpha k}\epsilon_0$$

holds and (ii) *$\theta^k(u)_{\theta^k}$ satisfies $|\theta^k(u)_{\theta^k}| \leq 1$, then we have*

$$\operatorname{osc}(u,p,\theta^{k+1}) \leq \theta^{2\alpha(k+1)}\epsilon_0\,, \tag{49}$$

$$|\theta^{k+1}(u)_{\theta^{k+1}}| \leq 1\,. \tag{50}$$

Proof: Let us consider the rescaled functions

$$U(x,t) \;=\; \frac{u(\theta^k x, \theta^{2k}t) - (u)_{\theta^k}}{\theta^{2\alpha k}}\,, \qquad P(x,t) \;=\; \frac{p(\theta^k x, \theta^{2k}t) - (p)_{\theta^k}(\theta^{2k}t)}{\theta^{(2\alpha-1)k}}\,.$$

Then we have

$$⨍_{Q_1} U \;=\; ⨍_{B(0,1)} P(\cdot, \theta^{2k}t) \;=\; 0\,.$$

By the assumption $\operatorname{osc}(u,p,\theta^k) \leq \theta^{2\alpha k}\epsilon_0$ we also have

$$\left(⨍_{Q_1} |U|^3\right)^{\frac{1}{3}} + \left(⨍_{Q_1} |P|^{\frac{3}{2}}\right)^{\frac{2}{3}} \leq \epsilon_0\,.$$

Note that (U,P) satisfies (NSEdrift-ϵ_k) in the Step (ii) of the proof of Lemma 2.3 replacing (v_k, q_k), ϵ_k, and b respectively by (U,P), $\theta^{k+2\alpha k}$, and $\theta^k(u)_{\theta^k}$. Hence, by reproducing a similar argument as in the proof of Lemma 2.3, we can prove that

$$\operatorname{osc}(U,P,\theta) \leq \theta^{2\alpha}\epsilon_0\,,$$

which implies the first assertion (49). Moreover, from $|\theta^k(u)_{\theta^k}| \leq 1$ we see that

$$\begin{aligned}
|\theta^{k+1}(u)_{\theta^{k+1}}| &\leq \theta^{k+1}|(u)_{\theta^{k+1}} - (u)_{\theta^k}| + \theta|\theta^k(u)_{\theta^k}| \\
&\leq \theta^{k+1} ⨍_{Q_{\theta^{k+1}}} |u - (u)_{\theta^k}| + \frac{1}{2} \\
&\leq \theta^{k+1}\theta^{-5}\left(⨍_{Q_{\theta^k}} |u - (u)_{\theta^k}|^3\right)^{\frac{1}{3}} + \frac{1}{2} \\
&\leq \theta^{k+1}\theta^{-5}\theta^{2\alpha k}\epsilon_0 + \frac{1}{2} \\
&\leq 1\,,
\end{aligned}$$

where we have used the conditions $\epsilon_0 \in (0, \frac{\theta^5}{2}]$ and $\theta \in (0, \frac{1}{2})$. Then we obtain the second assertion (50). The proof is complete. □

2.4 Comments

In this subsection we make comments on the regularity of the solutions to (NSE).

Regularity
We refer to Serrin [16], Struwe [18], Ladyzhenskaya and Seregin [10], Prodi, Takahashi [19], and Escauriaza, Seregin, and Sverák [5]. Let (u, p) be a solution to (NSE) such that

$$u \in L_t^\infty L_x^2(Q_1)\,, \quad \nabla u \in L_t^2 L_x^2(Q_1)\,.$$

Then if additionally u satisfies

$$u \in L_t^p L_x^q(Q_1) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 1\,, \quad p \in [2,\infty)\,,\ q \in (3,\infty)\,,$$

then we have for all fixed $t \in (-1, 0)$,

$$u(\cdot, t) \in C^\infty(B(0,1))\,.$$

A more quantitative result is available due to Necas, Ruzicka, and Sverák [13]: let ϵ_* be the constant in (37) in the rescaled version of Theorem 2.2. Then for all $k \in \mathbb{N}$, there exists a constant $C_k \in (0,\infty)$ depending on k such that for all $r \in (0,\infty)$ and for all LSWS (u, p) in Q_r, if

$$\frac{1}{r^2}\int_{Q_r} |u|^3 + |p|^{\frac{3}{2}} \le \epsilon_*$$

holds, then we have $\nabla^k u \in C^\alpha_{\text{par}}(\overline{Q_{\frac{r}{2}}})$ and

$$\sup_{(x,t)\in Q_{\frac{r}{2}}} |\nabla^k u| \le C_k r^{-1-k}\,.$$

Partial regularity
We state a theorem without proof.

Table 1: **History**

Leray-Hopf	LSWS, ϵ-regularity
Leray 1934 [11], Hopf 1951 [8]	Scheffer 1976-77 [14, 15]
$\mathbb{R}^3$ Calderon-Zygmund theory, $p \in L^{\frac{5}{3}}_{t,x}$	Caffarelli, Kohn, Nirenberg 1982 [4]
Sohr and Von Wahl 1986 [17]	$p \in L^{\frac{5}{4}}_{t,x}(Q_1)$
bounded or exterior domain,	Lin 1998 [12]
$p \in L^{\frac{5}{3}}_{t,x}$ for smooth initial data	$p \in L^{\frac{3}{2}}_{t,x}(Q_1)$
	Ladyzhenskaya and Seregin 1999 [10]
	bounded domain Ω,
	$p \in L^{\frac{3}{2}}(\Omega \times (\delta, T))$ for $\delta > 0$
	Vasseur 2007 [20], Kukavica 2009 [9]

Theorem 2.5 (limsup criteria) *Let ϵ_* be the constant in* (37) *in the rescaled version of Theorem 2.2. Then there exists a constant $\epsilon_1 \in (0,\infty)$ such that for all LSWS (u,p) to* (NSE) *in Q_1, if*

$$\limsup_{r\to\infty} \frac{1}{r}\int_{Q_r} |\nabla u|^2 \le \epsilon_1$$

holds, then we have

$$\frac{1}{\rho^2}\int_{Q_\rho} |u|^3 + |p|^{\frac{3}{2}} \le \epsilon_*$$

for some $\rho \in (0,\infty)$. Thus the rescaled version of Theorem 2.2 implies $u \in C^{\alpha}_{\mathrm{par}}(\overline{Q_{\frac{\rho}{2}}})$.

We briefly describe an important application of the limsup criteria. Let (u,p) be an LSWS to (NSE) in Q_1. Then the point $(x,t) \in Q_1$ is said to be regular for $u = u(x,t)$ if

$$u \in L^\infty(Q_r(x,t)) \text{ for some } r \in (0,1)\,,$$

and is said to be singular for $u(x,t)$ if

$$u \notin L^\infty(Q_r(x,t)) \text{ for any } r \in (0,1)\,.$$

The singular set $S \subset Q_1$ of $u(x,t)$ is defined by

$$S = \{(x,t) \in Q_1 \mid u \text{ is singular at } (x,t)\}\,.$$

Then we can prove the following statement by using Theorem 2.5:

$$\mathcal{H}^1_{\mathrm{par}}(S) = 0\,, \qquad \dim_{\mathrm{haus}}(S) \le 1\,.$$

Here $\mathcal{H}^1_{\mathrm{par}}$ denotes the parabolic Hausdorff measure of S and $\dim_{\mathrm{haus}}(S)$ denotes the parabolic Hausdorff dimension of S.

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