GLOBALIZATION OF SUPERCUSPIDAL REPRESENTATIONS OVER FUNCTION FIELDS AND APPLICATIONS

WEE TECK GAN

Kyoto University / Distinguished Visiting Project Professor

National University of Singapore / Professor

Abstract. There notes are taken by Shuji Horinaga, Kaoru Sano, and Miyu Suzuki. They are based on KTGU Special Lectures delivered at Department of Mathematics, Kyoto University in December 2015 and January 2016.

Contents

1. Introduction

Let k be a global field, i.e., either a number field like $\mathbb{Q}$ or a function field like $\mathbb{F}_p(t)$. Here, a function field means a function field of an absolutely irreducible smooth projective curve over a finite field $\mathbb{F}_q$. We denote by $\mathbb{A}$ the adèle ring of k. Take a connected reductive group H defined over k, e.g., $H = \mathrm{GL}_n$. Let $Z = Z(H)^\circ$ be the identity component of the center of H, which is a central torus. The set of adèle points $H(\mathbb{A})$ forms a locally compact group and it contains the set of k-rational points $H(k)$ as a discrete subgroup. It is similar to the case of $\mathbb{R}$ containing $\mathbb{Z}$. In this situation, the quotient space

$$[H] = Z(\mathbb{A})H(k)\backslash H(\mathbb{A})$$

is of finite volume with respect to an invariant measure.

A function

$$f\colon H(k)\backslash H(\mathbb{A}) \longrightarrow \mathbb{C} \tag{1.1}$$

satisfying certain smoothness, finiteness, and growth conditions is called an *automorphic form* on H. We denote by $\mathcal{A}(H)$ the space of automorphic forms on H. The group $H(\mathbb{A})$ acts on $\mathcal{A}(H)$ by right translation. For a character

$$\omega\colon Z(k)\backslash Z(\mathbb{A}) \longrightarrow \mathbb{C}^\times,$$

we denote by $\mathcal{A}_\omega(H)$ the subspace of $\mathcal{A}(H)$ consisting of automorphic forms which satisfy

$$f(hz) = \omega(z)f(h) \tag{1.2}$$

for all $h \in H(\mathbb{A})$ and $z \in Z(\mathbb{A})$. In this case, we call ω the *central character*.

Let ω be a unitary character. We say that a function f satisfying (1.1) and (1.2) is an L^2-*function* if the inequality

$$\int_{[H]} |f|^2 < \infty$$

holds. We denote the space of such functions by $L^2_\omega([H])$. We also consider the action of $H(\mathbb{A})$ on this space by right translation.

For a parabolic k-subgroup P of H and its Levi decomposition $P = MN$, we define the *constant term* of f along N by

$$f_N(h) = \int_{[N]} f(nh)\,\mathrm{d}n.$$

Here, the integral is taken over the compact quotient $[N] = N(k)\backslash N(\mathbb{A})$.

Definition 1.1. An automorphic form $f \in \mathcal{A}(H)$ is a *cusp form* if the constant term f_N is zero for all proper parabolic k-subgroup P with Levi decomposition $P = MN$.

We denote the space of cusp forms with central character ω by $\mathcal{A}_{\omega,\mathrm{cusp}}(H)$. It is contained in the intersection

$$\mathcal{A}^2_\omega(H) = \mathcal{A}_\omega(H) \cap L^2_\omega([H]).$$

Let $L^2_{\omega,\mathrm{cusp}}([H])$ be the L^2-closure of $\mathcal{A}_{\omega,\mathrm{cusp}}(H)$. These subspaces are invariant under the action of $H(\mathbb{A})$ and in particular, these are semi-simple, i.e., decomposed into the direct sums of irreducible subrepresentations. Any irreducible summand of $L^2_{\omega,\mathrm{cusp}}([H])$ is called *a cuspidal representation*.

One of the goals in the *Global Langlands Program* is to classify all irreducible cuspidal representations of $H(\mathbb{A})$ in terms of *Galois representations* and some extra data. Thus our main objects of interest are cuspidal representations and cusp forms. However, the existence of non-zero cusp forms is not a priori clear form their definition.

Question 1.2. How to produce cusp forms?

We use the Poincaré series to answer this question. Let $C_c^\infty(H(\mathbb{A}))$ be the space of all smooth compactly supported functions on $H(\mathbb{A})$. Note that the space $C_c^\infty(H(\mathbb{A}))$ can be identified with the restricted tensor product $\bigotimes'_v C_c^\infty(H(k_v))$ of analogous spaces of functions on $H(k_v)$, where v runs over all places of k. Take a non-zero element f in $C_c^\infty(H(\mathbb{A}))$. By the above remark, a function f can be decomposed into a linear combination of the tensor product of local functions $\otimes'_v f_v$ and for almost all v, a function f_v is the characteristic function of a maximal compact subgroup of $H(k_v)$. Set $\mathcal{P}(f)$ be the function on $H(k)\backslash H(\mathbb{A})$ given by

$$\mathcal{P}(f)(h) = \sum_{\gamma \in H(k)} f(\gamma h)$$

for $h \in H(\mathbb{A})$. We call the function $\mathcal{P}(f)$ the *Poincaré series*.

We need to show this sum converges for every $h \in H(\mathbb{A})$. For a fixed $h \in H(\mathbb{A})$, we only need to sum over $\gamma \in H(k) \cap \mathrm{Supp}(f)h^{-1}$. This intersection is in fact a finite set since this is discrete and compact in $H(\mathbb{A})$. Therefore the above sum converges absolutely and moreover, a function $\mathcal{P}(f)$ is a smooth function. There remains to show that we can take a function f so that the Poincaré series $\mathcal{P}(f)$ is a non-zero cusp form. Take a function $f = \otimes_v f_v \in C_c^\infty(H(\mathbb{A}))$. Pick a non-archimedean place v_0 of k. Fix functions f_v for $v \neq v_0$ such that $f_v(1) \neq 0$. Let f_{v_0} be the characteristic function of a compact open neighborhood K_{v_0} of the identity of $H(k_{v_0})$.

Lemma 1.3. We can take a sufficiently small compact open neighborhood K_{v_0} so that $H(k) \cap \mathrm{Supp}(f) = \{1\}$ holds.

For K_{v_0} in Lemma 1.3, we have $\mathcal{P}(f)(1) \neq 0$.

To construct a function $f = \otimes'_v f_v$ such that the Poincaré series $\mathcal{P}(f)$ is a cusp form, we need to introduce the local analogue of the notion of cusp forms.

Definition 1.4. For a representation (π, V_π) of $H(k_v)$ and a proper parabolic k_v-subgroup $P = MN$ of H, let π_N be the quotient of V_π by the subspace of V_π spanned by elements of the form $\pi(n)w - w$ with $n \in N$ and $w \in V_\pi$. A representation π is called *supercuspidal* if $\pi_N = 0$ for every proper parabolic k_v-subgroup P.

For an irreducible admissible representation (π, V_π) of $H(k_v)$, the following are equivalent:

(i) π is supercuspidal.
(ii) π is not a subquotient of any representations obtained by parabolic induction.
(iii) π is not a submodule of any representations obtained by parabolic induction.
(iv) All matrix coefficients of π are compactly supported in $H(k_v)$ modulo $Z(k_v)$.

Here we explain the definition of matrix coefficients. Let $\pi^\vee$ denote the contragredient representation of π and denote the canonical pairing by $\langle\ ,\ \rangle$. Consider the linear map

$$\pi \otimes \pi^\vee \longrightarrow C^\infty(H(k_v)) \tag{1.3}$$

$$w \otimes w^\vee \longmapsto f_{w,w^\vee}, \tag{1.4}$$

where $f_{w,w^\vee}$ is defined by $h \mapsto \langle \pi(h)w, w^\vee \rangle$. This map is $H(k_v) \times H(k_v)$-equivariant if we consider the action of this group on $C^\infty(H(k_v))$ by translation: the first factor acts by right translation and the second by left translation. A function of the form $f_{w,w^\vee}$ is called a *matrix coefficient* of π.

Supercuspidal representations are the most fundamental parts of irreducible representations since these representations cannot be obtained by parabolic induction. However, existence of supercuspidal representations is not clear.

Question 1.5. Does there exist an irreducible supercuspidal representation?

If the place v is archimedean, then there are no supercuspidal representations. Otherwise they do exist but there is no easy way to construct. One of the known methods of construction is so called *reduction mod* v. We will explain only the idea. For simplicity, we just consider the case $H(k_v) = \mathrm{PGL}_2(\mathbb{Q}_p)$. This group contains $K = \mathrm{PGL}_2(\mathbb{Z}_p)$ as a maximal compact subgroup. There is a surjective homomorphism from K to the finite group $\mathrm{PGL}_2(\mathbb{F}_p)$, which is called a *reduction mod* p *map*. The notion of (super)cuspidality of the representations of $\mathrm{PGL}_2(\mathbb{F}_p)$ is similarly defined. For a supercuspidal representation τ of $\mathrm{PGL}_2(\mathbb{F}_p)$, we denote its pullback to K by the same symbol. Then the representation $\pi = \mathrm{ind}_K^{H(k_v)}(\tau)$, the compactly induced representation from τ, is supercuspidal. More generally, there exists a construction of cuspidal representations of reductive groups over finite fields by Deligne-Lusztig, and hence this construction works well. Supercuspidal representations obtained by this way is called *depth zero*. In general, supercuspidal representations are not exhausted by such construction.

For simplicity, let us assume that H is semisimple. Fix a non-archimedean place v_0 and an irreducible supercuspidal representation π_{v_0} of $H(k_{v_0})$. Let f_{v_0} be a non-zero matrix coefficient of π_{v_0} which is an element of $C_c^\infty(H(k_{v_0}))$ by the assumption of semisimplicity.

Proposition 1.6. Under the above assumptions, the Poincaré series $\mathcal{P}(f)$ is a cusp form.

Proof. Let $f^{v_0} = \otimes'_{v\neq v_0} f_v$. For any proper parabolic k-subgroup $P = MN$ of H, consider the linear functional l_N on π_{v_0} given by

$$l_N(\varphi) = \int_{[N]} \mathcal{P}(f_\varphi \otimes f^{v_0})\, \mathrm{d}n,$$

where a function $f_\varphi(h) = \langle \pi_{v_0}(h)\varphi, w^\vee \rangle$ is a matrix coefficient of π_{v_0} attached to $\varphi \in \pi_{v_0}$ and a fixed non-zero element $w^\vee \in \pi_{v_0}^\vee$. Clearly the linear functional l_N factors through $(\pi_{v_0})_N$. On the other hand, we have $(\pi_{v_0})_N = 0$ since π_{v_0} is supercuspidal. Hence the linear functional l_N vanishes and the Poincaré series $\mathcal{P}(f)$ is a cusp form. □

To summarize, if you want to construct a non-zero cusp form, what you need to do is as follows. Fix two non-archimedean places v_0 and v_1 of k and at each place v, take a non-zero smooth function f_v which satisfies the following conditions:

- if $v = v_0$, a function f_{v_0} is a non-zero matrix coefficient of a supercuspidal representation,
- if $v = v_1$, a function f_{v_1} is a characteristic function of a sufficiently small compact open neighborhood K_{v_1} of the identity of $H(k_{v_1})$, and
- if v is other than v_0, v_1, a function f_v is an arbitrary non-zero smooth function.

Under the above assumptions, the Poincaré series $\mathcal{P}(\otimes_v f)$ is a non-zero cusp form.

Remark 1.7. Consider the irreducible summands of the submodule of $\mathcal{A}_{\mathrm{cusp}}(H)$ generated by $\mathcal{P}(f)$. One of them, say Π, provides us a globalization of the local representation π_{v_0}. In the above argument, we take a test function supported on a sufficiently small neighborhood of the identity at one place v_1. Therefore we *lose the control* of the local component Π_{v_1}.

2. The main theorem

In this section, we state our main theorem and some remarks about it.

Theorem 2.1. Suppose that the following data are given:

- $k = k(Y)$ a global function field (k is the function field of an absolutely irreducible curve Y over $\mathbb{F}_q$),
- $S_0 \neq \phi$ a finite set of places of k,

- H a connected reductive group over k,
- Z the identity component of the center of H,
- $N \subset H$ smooth connected k-split unipotent subgroup,
- χ an automorphic character of $N(\mathbb{A})$, i.e.,

$$\chi = \prod_v \chi_v \colon N(k)\backslash N(\mathbb{A}) \longrightarrow \mathbb{C}^\times,$$

- $\omega\colon Z(k)\backslash Z(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ a unitary character,
- π_{v_0} a supercuspidal representation of $H(k_{v_0})$ for every $v_0 \in S_0$ such that

$$\mathrm{Hom}_{Z(k_{v_0})N(k_{v_0})}(\pi_{v_0}, \omega_{v_0} \otimes \chi_{v_0}) \neq 0,$$

  i.e., a representation π_{v_0} has a central character ω_{v_0} and π_{v_0} is $(N(k_{v_0}), \chi_{v_0})$-*distinguished*.

Then there exists a cuspidal representation $\Pi = \bigotimes'_v \Pi_v$ such that

(1) for every $v_0 \in S_0$, we have $\Pi_{v_0} \cong \pi_{v_0}$,
(2) for every $v \notin S_0$, a local component Π_v is a submodule of $\mathrm{Ind}_{P_v(k_v)}^{H(k_v)}(\tau_v)$ with $P_v \subset H \times_k k_v$ a minimal parabolic subgroup over k_v and $\Pi_v|_{H_v^{\mathrm{der}}}$ has depth 0, and,
(3) Π has the central character ω and Π is *globally distinguished by* (N, χ), i.e.,

$$\int_{[N]} \chi(n)^{-1}\varphi(n)\,\mathrm{d}n \neq 0$$

for some $\varphi \in \Pi$.

Note that the map

$$\Pi \ni \varphi \mapsto \int_{[N]} \chi(n)^{-1}\varphi(n)\,\mathrm{d}n \in \mathbb{C}$$

is an element of $\mathrm{Hom}_{N(\mathbb{A})}(\Pi, \chi)$.

Example 2.2. Let us see two special cases for later use.

- N is trivial.
- H is quasi-split (i.e., it has a Borel subgroup B defined over k), N is the unipotent radical of B and χ is a *generic character* on N.

This observation only provides an element in the space of cuspidal automorphic forms. Of course once we obtain a non-zero vector in the representation space, we can construct a submodule generated by it. However, what we want to know precisely is the irreducible summands of that submodule and those local components.

Remark 2.3. There are some comments about our main theorem.

(1) We can prove exactly the same theorem in number field case by the similar argument as in the Introduction. The most different point is that function field does not have archimedean places which we can "sacrifice." Therefore we should be more careful than number field case.

(2) When talking about unipotent groups, unlike characteristic 0 case, we need to notice it is assumed to be smooth, connected and k-split. A unipotent group U over k is called k-split if it is a successive extension of $\mathbb{G}_a$ over k. If k is perfect, then any unipotent group is k-split.

Example 2.4 (Rosenlicht)**.** Assume that k is not perfect and characteristic $p > 0$. Take $a \in k \setminus k^p$. Set $U = \{y^p = x - ax^p\} \subset \mathbb{G}_a^2$. This is a unipotent k-subgroup of $\mathbb{G}_a^2$.

Over $L = k(a^{\frac{1}{p}})$, the group U is isomorphic to $\mathbb{G}_a$ (as algebraic groups) by the map $(x, y) \mapsto y + a^{\frac{1}{p}} x$. However, over k, U is not isomorphic to $\mathbb{G}_a$ as a scheme. See [2, Appendix B] and [6, Chapter V].

Definition 2.5. A unipotent k-group U is *k-wound* if every k-morphism of scheme $\mathbb{G}_a \longrightarrow U$ is constant.

The unipotent group U in the above example is k-wound. If k is a local function field, then a unipotent group U is k-wound if and only if $U(k)$ is compact in the analytic topology. This is an analogue of “anisotropic torus”.

Example 2.6. Notation are as above. Let $G = \operatorname{Res}_{L/k} \mathbb{G}_m$. Here, Res is the Weil restriction of scalars. Then for a k-algebra R, the set of R-valued points of G is $(R \otimes_k L)^\times$ and there is a natural embedding $\mathbb{G}_m \hookrightarrow G$.

Set $H = G/\mathbb{G}_m$. Each element in $H(k) = L^\times / k^\times$ has order dividing p. Hence H is isomorphic to $\mathbb{G}_a^{p-1}$ over L but k-wound.

Corollary 2.7. Let F be a local field of characteristic $p > 0$. Suppose the following data are given:

(i) a central torus $Z_F \subset H_F$,
(ii) a parabolic subgroup $P_F = M_F N_F$,
(iii) a character $\omega_F \colon Z_F \longrightarrow S^1$,
(iv) a character $\chi_F \colon N_F \longrightarrow S^1$ such that the M_F-orbit of χ_F is open,
(v) $\pi_1, \ldots, \pi_a$ supercuspidal representations of H_F which are (N_F, χ_F)-distinguished with central character ω_F.

Then there are

- a global field k with places $v_1, \ldots, v_a$ such that $k_{v_i} \cong F$,
- a central torus $Z \subset H$ over k globalizing (i) at each v_i,
- a parabolic subgroup $P = MN \subset H$ over k globalizing (ii) at each v_i,
- a character $\omega \colon Z(k)\backslash Z(\mathbb{A}) \longrightarrow S^1$ globalizing (iii),
- a character $\chi \colon N(k)\backslash N(\mathbb{A}) \longrightarrow S^1$ such that χ_{v_i} and χ_F are in the same M_F-orbit,
- a cuspidal representation Π of $H(\mathbb{A})$ as in the theorem.

3. Proof of the main theorem

In this section, we will prove the main theorem. We assume that a group H is a split semisimple group, a subgroup $B = TU$ of H is a Borel subgroup of H and N is the trivial group or a unipotent subgroup of B. Fix an embedding $\iota\colon H \hookrightarrow \mathrm{SL}_n \subset \mathrm{GL}_n$. We then get coordinate functions $x_{i,j}$ on H. By this embedding ι, we recognize H as a subgroup of GL_n. Without loss of generality, we may assume the following conditions:

- $B = H \cap$ (the upper triangular subgroup in SL_n),
- $T = H \cap$ (the diagonal subgroup in SL_n),
- $U = H \cap$ (the upper triangular unipotent subgroup in SL_n),
- $\overline{U} = H \cap$ (the lower triangular unipotent subgroup in SL_n), and,
- $N \subset U$.

Let $\mathcal{O}_{S_0}$ be the ring of S_0-integers in k. Then there exists an $\mathcal{O}_{S_0}$-integral structure on GL_n. Hence, we have an $\mathcal{O}_{S_0}$-integral structure on H.

Let S be a large finite set of places which satisfies

- $S \cap S_0 = \emptyset$,
- a group H is a smooth reductive group over $\mathcal{O}_v$ for any $v \notin S \cap S_0$,
- a subgroup I_v^+ is an Iwahori subgroup of $H(k_v)$,
- a subgroup I_v^- is an lower Iwahori subgroup of $H(k_v)$, and,
- a group N is smooth over $\mathcal{O}_v$ and χ_v is trivial on $N(\mathcal{O}_v)$.

Here, the subgroup I_v^+ (resp. I_v^-) is the intersection of $H(k_v)$ and the Iwahori subgroup of GL_n (resp. the lower Iwahori subgroup of GL_n).

First, we define an open compact subgroup $C = \prod_v C_v$ as follows. Fix an arbitrary open compact subgroup $C_{S_0} = \prod_{v\in S_0} C_v$ of $\prod_{v\in S_0} H(k_v)$. Fix two finite sets S_1 and S_2 of places which are disjoint from $S_0 \cap S$. Define the open subgroups C_v for each place $v \notin S_0$ by

- for $v \notin S \cup S_0 \cup S_1 \cup S_2$, let $C_v = H(\mathcal{O}_v)$,
- for $v \in S_1$, let $C_v = I_v^+$,
- for $v \in S_2$, let $C_v = I_v^-$, and,
- for $v \in S$, a subgroup C_v is an Iwahori subgroup so that χ_v is trivial on C_v.

Lemma 3.1. If S_1 and S_2 are sufficiently large, we have

$$H(k) \cap C \subset N(k).$$

Proof. It is sufficient to prove that if S_1 and S_2 are sufficiently large, the intersection $H(k) \cap C$ is trivial. For every S_1 and S_2, the intersection $H(k) \cap C$ is a finite set, since $H(k)$ is a discrete subgroup of $H(\mathbb{A})$ and C is an compact subgroup of $H(\mathbb{A})$. Take an element $\gamma \in H(k) \cap N(k)$. Let $T = S \cap S_0 \cap S_1 \cap S_2$. Then $x_{i,j}(\gamma) \in \mathcal{O}_v$, since $\prod_{v\notin T} C_v = \prod_{v\notin T} H(\mathcal{O}_v)$. This means that $x_{i,j}(\gamma)$ has no pole at $v \in |Y|$

such that $v \notin T$. Then there exists M such that $\mathrm{ord}_v(x_{i,j}) < M$ for all i, j and $v \in S \cap S_0$. For $v \in S_1$, we have $C_v = I_v^+$. Hence, coordinates $x_{i,j}(\gamma)$ vanish at all $v \in S_1$ for $i > j$. Similarly, $x_{i,j}(\gamma)$ vanishes at all $v \in S_2$ for $i < j$. Therefore, if S_1 and S_2 are sufficiently large, we have

$$x_{i,j}(\gamma) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

This completes the proof. □

We replace the finite sets S_1 and S_2 with the sets as in Lemma 3.1.

Next, we will construct the test function.

Lemma 3.2. If π_v is a supercuspidal representation of $H(k_v)$ so that

$$\mathrm{Hom}_{N(k_v)}(\pi, \chi_v) \neq 0,$$

then we can find a matrix coefficient f_v of π_v which satisfies

$$\int_{N(k_v)} f_v(n)\overline{\chi_v}(n)\,\mathrm{d}n \neq 0.$$

Proof. Choose a nonzero functional $l \in \mathrm{Hom}_{N(k)}(\pi_v, \chi_v)$ and fix a nonzero element $w_0 \in \pi_v$. Then we have $\pi_v = C_c^\infty(H(k)) \cdot w_0$, where

$$\pi_v(f_v)w_0 = \int_{H(k_v)} f_v(h)\pi_v(h)w_0\,\mathrm{d}h$$

for $f_v \in C_c^\infty(H(k))$. Hence there is $f_v \in C_c^\infty(H(k))$ such that $l(\pi_v(f_v)w_0) \neq 0$. On the other hand,

$$\begin{aligned} l(\pi_v(f_v)w_0) &= \int_{H(k_v)} f_v(h)l(\pi_v(h)w_0)\,\mathrm{d}h \\ &= \int_{N(k_v)\backslash H(k_v)} \left(\int_{N(k)} f_v(nh)l(\pi_v(n)\pi_v(h)w_0)\,\mathrm{d}n \right) \mathrm{d}h \\ &= \int_{N(k_v)\backslash H(k_v)} l(\pi_v(hw_0)) \left(\int_{N(k_v)} \chi(n)f_v(nh)\,\mathrm{d}n \right) \mathrm{d}h. \end{aligned}$$

Now we get $f_v \in C_c^\infty(H(k))$ with non-vanishing integral

$$\int_{N(k_v)} \chi(n)f_v(nh)\,\mathrm{d}n \neq 0.$$

Since π_v is supercuspidal, the map

$$C_c^\infty(H(k_v)) \longrightarrow \mathrm{End}(V_{\pi_v}) \cong \pi_v \otimes \pi_v^\vee$$

defined by $f_v \mapsto \pi_v(f_v)$ has a splitting, given by

$$w \otimes w^\vee \mapsto f_{w,w^\vee}.$$

This splitting leads us to the following decomposition of $C_c^\infty(H(k_v))$ as an $H(k_v)\times H(k_v)$-module: we have a decomposition

$$C_c^\infty(H(k_v)) = (\pi_v \otimes \pi_v^\vee) \oplus C',$$

where C' is a submodule which has no subquotient isomorphic to $\pi_v \otimes \pi_v^\vee$. The map $f \mapsto \pi_v(f_v)$ is zero on C' and hence we may assume f_v above is of the form

$$f_v(h) = \langle \pi_v^\vee(h)w^\vee, w\rangle$$

for some w and $w^\vee$. By the previous argument, we have

$$\begin{aligned} 0 &\neq \int_{N(k_v)} \chi_v(n)\langle \pi_v^\vee(nh)w^\vee, w\rangle \,\mathrm{d}n \\ &= \int_{N(k_v)} \chi_v(n)\langle \pi^\vee(h)w^\vee, \pi(n^{-1})w\rangle \,\mathrm{d}n \end{aligned}$$

for some $h \in H(k_v)$. We set $w^\vee = \pi_v^\vee(h)w$. Then $f = f_{w,w^\vee}$ satisfies the desired property. This completes the proof. □

Take a matrix coefficient f_v for $v \in S_0$ as in Lemma 3.2. Set $C_v = \mathrm{Supp}(f_v)$ for $v \in S_0$. For $v \notin S_0$, let $f_v = 1_{C_v}$. We then define a test function f by $f = \otimes_v f_v$.

Proof of the main theorem. Let $\mathcal{P}(f)$ be the Poincare series defined by

$$\mathcal{P}(f)(h) = \sum_{\gamma\in H(k)} f(\gamma h), \qquad h \in H(\mathbb{A}).$$

We claim that

$$\int_{N(k)\backslash N(\mathbb{A})} \overline{\chi}(n)\mathcal{P}(f)(n)\,\mathrm{d}n \neq 0.$$

Indeed, we have

$$\begin{aligned} \int_{N(k)\backslash N(\mathbb{A})} \chi^{-1}(n) \sum_{\gamma\in N(k)} f(\gamma n)\,\mathrm{d}n &= \sum_{\gamma\in N(k)} \int_{N(k)\backslash N(\mathbb{A})} \chi^{-1}(n) f(\gamma n)\,\mathrm{d}n \\ &= \int_{N(\mathbb{A})} \chi^{-1}(n) f(n)\,\mathrm{d}n \neq 0. \end{aligned}$$

Note that the convergence of the integral comes from Lemma 3.1.

Therefore, $\mathcal{P}(f)$ is cuspidal and (N,χ)-distinguished. Hence we see

$$\mathcal{P}(f) \in L^2_{\mathrm{cusp}}(H(k)\backslash H(\mathbb{A})).$$

Let Π be an irreducible summand of the representation generated by $\mathcal{P}(f)$ on $H(\mathbb{A})$. This gives Π as in the main theorem. This completes the proof. □

4. Proof of the globalization theorem

First we globalize the local fields. Let F be an local field of characteristic $p > 0$. Then, there exists q such that $F \cong \mathbb{F}_q((T))$. Let $k_0 = \mathbb{F}_q(T)$. By Krasner's Lemma, we have the globalization of local field.

Lemma 4.1. Given finite Galois extension E/F of local fields, we can fined a Galois extension k'/k with $k_v \cong F$ which satisfies $k' \otimes_k k_v = E$ and $[k' \colon k] = [E \colon F]$.

Next, we globalize a parabolic subgroup $H_F \supseteq P_F$. We assume first that the group H_F is quasi-split. Let H_s be a split group over $\mathbb{Z}$ so that $H_s \times_{\mathbb{Z}} \overline{F} \cong H_F \times_F \overline{F}$. Then,

$$\left\{ \begin{array}{l} \text{quasi-split reductive groups over } F \\ \text{which are isomorphic to } H_s \text{ over } F \end{array} \right\} \cong H^1(F, \mathrm{Out}(H_s)).$$

For an isomorphic class $[H_F]$ of such reductive groups, we denote by $[C_{H_F}]$ the corresponding cocycle in $H^1(F, \mathrm{Out}(H_s))$. If E/F is a finite Galois extension such that H_F is split, then

$$C_{H_F} \colon \mathrm{Gal}(E/F) \longrightarrow \mathrm{Out}(H_s).$$

By Lemma 4.1, we have an isomorphism $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(k'/k)$. Hence, we have a cocycle

$$c \colon \mathrm{Gal}(k'/k) \longrightarrow \mathrm{Out}(H_s).$$

Consider a cocyle $[c] \in H^1(k, \mathrm{Out}(H_s))$, we have a quasi-split group H over k such that $H \times k_v \cong H_F$. Let

$$\Psi = (X^*(T_s), \triangle(T_s, B_s), X_*(T_s), \triangle^\vee)$$

be the root datum corresponding to a Borel subgroup $T_s U_s = B_s \subset H_s$. Then we obtain a $\mathrm{Gal}(E/F)$-twisted root datum corresponding to such a $[H]$ or $[c]$. Hence, we have a Borel subgroup of H, moreover, we get a globalization of $P_F \subset H_F$.

We now consider the general case. Given $H_F \supseteq P_F$. Let H'_F be the quasi-split inner form of H_F. By the above argument, we can globalize a parabolic subgroup $P'_F \subset H'_F$. Let $P' \subset H'$ be such a globalization. For $[H_F]$, we obtain a cocycle

$$[C_{H_F}] \in H^1(F, \mathrm{Int}(H'_F)) = H^1(F, H'_{F,\mathrm{Ad}}).$$

We need to show that the natural localization map

$$H^1(k, H'_{\mathrm{Ad}}) \longrightarrow H^1(F, H'_{F,\mathrm{Ad}})$$

is surjective. This is already proved by Borel-Harder in characteristic 0 and Thang-Tan in positive characteristics. We need to consider

$$\mathrm{Inn}(H' \supset P') \cong P'_{\mathrm{Ad}} \subset H'_{\mathrm{Ad}}.$$

Let M' and M'_F be Levi subgroups of P' and P'_F, respectively. Then, we need to show subjectivity of the localization map

$$H^1(k, M'_{\mathrm{Ad}}) = H^1(k, P'_{\mathrm{Ad}}) \longrightarrow H^1(F, P'_{F,\mathrm{Ad}}) = H^1(F, M'_{\mathrm{Ad}}).$$

There exists an exact sequence

$$(M'_{\mathrm{Ad}})^{\mathrm{der}} \longrightarrow M'_{\mathrm{Ad}} \longrightarrow A \longrightarrow 1,$$

where A is the split torus of M'. Then, we have a diagram

$$\begin{array}{ccccc} H^1(k, M'^{\mathrm{der}}_{\mathrm{Ad}}) & \twoheadrightarrow & H^1(k, M'_{\mathrm{Ad}}) & \twoheadrightarrow & H^1(A) = 0 \\ \downarrow\!\!\downarrow & & \downarrow & & \\ H^1(F, M'^{\mathrm{der}}_{\mathrm{Ad}}) & \twoheadrightarrow & H^1(F, M'_{\mathrm{Ad}}) & \longrightarrow & 0. \end{array}$$

This forces that $H^1(k, M'_{\mathrm{Ad}}) \longrightarrow H^1(F, M'_{\mathrm{Ad}})$ is surjective. Hence, we can globalize a parabolic subgroup $P_F \subset H_F$ for general H_F.

Finally, we globalize the central character ω.

Lemma 4.2. (D. Prasad) Let k be a global field of characteristic $p > 0$ with $k_{v_0} = F$. Let Z be a k-torus such that F-rank r of $Z_F = Z_{v_0}$ is equal to k-rank of Z. For a given unitary character $\omega_F \colon Z_F \longrightarrow S^1$, there is a global automorphic character $\omega \colon Z(k)\backslash Z(\mathbb{A}) \longrightarrow S^1$ such that

- $\omega_{v_0} = \omega_F$
- ω is trivial on the compact subgroup
$$\Omega = \prod_{v \in T} Z(k_v)^1 \times \prod_{v \notin T \cup \{v_0\}} Z(k_v)^0,$$
with some non-empty finite set of places T.

Here, $Z(k_v)^0$ is the maximal compact subgroup of $Z(k_v)$ and $Z(k_v)^1$ is the maximal pro-p subgroup of $Z(k_v)^0$.

Proof. Consider the natural map

$$i \colon Z(k_{v_0})^0 \times \Omega \longrightarrow Z(k)\backslash Z(\mathbb{A}).$$

Since $Z(k_{v_0})^0 \times \Omega$ is compact, the image of i is a closed subgroup.

Claim: i is injective.

We assume this claim for a moment and deduce the assertion from it. By Pontrjagin theory, one can find a unitary character $\omega' \colon Z(k)\backslash Z(\mathbb{A}) \longrightarrow S^1$ which is trivial on Ω and whose restriction to $Z(k_{v_0})^0$ is equal to the restriction of ω_F.

Now we have

$$q \colon Z \longrightarrow \mathbb{G}_m^r$$

such that $\ker(q)$ is anisotropic over k_{v_0}. Then a homomorphism

$$\omega'_{v_0}\omega_F^{-1} \colon Z(k_{v_0}) \longrightarrow S^1$$

factors through

$$\mathrm{ord}_{v_0} \circ q \colon Z(k_{v_0}) \longrightarrow (k_{v_0}^\times)^r \longrightarrow \mathbb{Z}^r.$$

Note that $\mathrm{ord}_{v_0} \circ q$ has compact kernel and the image of finite index. By multiplying to ω' a character of the form $(\Pi_{i=1}^r |\cdot|_{\mathbb{A}}^{s_i}) \circ q$, we can make $\omega'_{v_0} = \omega_F$.

Next, we prove the claim. Let E be a splitting field of Z, which is a separable extension of k. Then $Z(k)$ can be embedded into $Z(E) \cong (E^{\times})^m$. We denote this embedding by $z \mapsto (z_1, \ldots, z_m)$ There are smooth projective curves Y and $\widetilde{Y}$ over $\mathbb{F}_q$ with function fields $k = \mathbb{F}_q(Y)$ and $E = \mathbb{F}_q(\widetilde{Y})$.

Take

$$z \in \ker(i) = Z(k) \cap (Z(k_{v_0})^0 \times \Omega).$$

Since this is in the maximal compact subgroup, each z_i corresponds to a constant function on $\widetilde{Y}$. On the other hand, z is in a maximal pro-p subgroup at some place and hence each z_i takes value 1 at some point. This means $z_i = 1$. □

5. Work of Vincent Lafforgue

In this section, let $k = \mathbb{F}_q(X)$ be a function field and G a connected reductive group over k. For simplicity, we assume that G is split and semi-simple.

5.1. Main theorems.

Theorem 5.1 (V. Lafforgue)**.** Let $\mathrm{Irr}_{\mathrm{cusp}}(G)$ be the set of cuspidal automorphic representations Π of $G(\mathbb{A})$, and let $\Phi(G)$ be the set of $\check{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of continuous semi-simple homomorphisms

$$\rho\colon \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow \check{G}(\overline{\mathbb{Q}}_\ell).$$

Then there is a map from $\mathrm{Irr}_{\mathrm{cusp}}(G)$ to $\Phi(G)$ such that for

$$\Pi = \bigotimes_v{}' \Pi_v \in \mathrm{Irr}_{\mathrm{cusp}}(G),$$

Π_v has Satake parameter $\rho_\Pi|_{\Gamma_{k,v}}$ at almost all v where Π_v is unramified,.

Definition 5.2. For a finite subscheme $N \subset X$, we define the *principal congruence subgroup* K_N of $G(\mathcal{O}) = \prod_v G(\mathcal{O}_v)$ by the kernel of reduction mod N map:

$$K_N = \ker(G(\mathcal{O}) \longrightarrow G(\mathcal{O}_N)).$$

Moreover, we define the space of K_N-invariant cusp forms by

$$\mathcal{A}_{\mathrm{cusp},N} = C_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A})/K_N),$$

which is of course contained in the space of cusp forms,

$$\mathcal{A}_{\mathrm{cusp}} = C_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A})) = \lim_{N} \mathcal{A}_{\mathrm{cusp},N}.$$

The latter space is a $G(\mathbb{A})$-module under right translation, but the former one is not stable under this action. However, $\mathcal{A}_{\mathrm{cusp},N}$ has an action of the Hecke algebra, which is defined as follows:

$$\mathcal{H}_N = C_c^\infty(K_N\backslash G(\mathbb{A})/K_N) = \bigotimes_v{}' C_c^\infty(K_{N,v}\backslash G(k_v)/K_{N,v}).$$

Understanding $\mathcal{A}_{\mathrm{cusp}}$ as a $G(\mathbb{A})$-module is actually "equivalent" to understand each $\mathcal{A}_{\mathrm{cusp},N}$ as an $\mathcal{H}_N$-module. Hence we treat $\mathcal{A}_{\mathrm{cusp},N}$ with fixed N.

Remark 5.3. If v is not in the support of N,

$$\mathcal{H}_{N,v} = C_c^\infty(K_{N,v}\backslash G(k_v)/K_{N,v})$$

is the usual spherical Hecke algebra.

Theorem 5.4 (V. Lafforgue)**.** For each fixed N, there is a decomposition of $\mathcal{H}_N$-modules

$$\mathcal{A}_{\mathrm{cusp},N} = \bigoplus_\rho \mathcal{A}_{N,\rho}$$

indexed by

$$\rho\colon\ \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow \check{G}(\overline{\mathbb{Q}}_\ell)$$

unramified outside N (modulo conjugacy) such that for any

$$\Pi = {\bigotimes_v}' \Pi_v \subset \mathcal{A}_{N,\rho}$$

and for all v outside N, Π_v has Satake parameter $\rho|_{\Gamma_{k_v}}$.

Remark 5.5. For fixed ρ, the submodule $\mathcal{A}_{N,\rho}$ consists q nearly equivalent representations. But it is a priori possible that for $\rho \neq \rho'$, the representations in $\mathcal{A}_{N,\rho}$ and $\mathcal{A}_{N,\rho'}$ are nearly equivalent.

If we have an operator on $\mathcal{A}_{\mathrm{cusp},N}$ which commutes with the $\mathcal{H}_N$-action, then we may obtain a decomposition of $\mathcal{A}_{\mathrm{cusp},N}$ into the generalized eigenspaces of this action.

Proposition 5.6 (Two key statements)**.** Notation is as above.

(i) For each N, there is a commutative subalgebra

$$B_N \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathcal{A}_{\mathrm{cusp},N})$$

which commutes with $\mathcal{H}_N$-action. Then there is a decomposition

$$\mathcal{A}_{\mathrm{cusp},N} = \bigoplus_\nu \mathcal{A}_{N,\nu}$$

indexed by characters $\nu\colon B_N \longrightarrow \overline{\mathbb{Q}}_\ell$. Moreover, for any v outside N, the action of $\mathcal{H}_{N,v}$ is realized by B_N.

(ii) There is a map

$$\mathrm{Hom}(B_N, \overline{\mathbb{Q}}_\ell) \longrightarrow \left\{\rho\colon \Gamma_k \longrightarrow \check{G}(\overline{\mathbb{Q}}_\ell)\right\}.$$

We denote the associate map with ν by ρ_ν.

Elements of B_N are of the form $S_{I,\gamma,f}$, where

- I is a finite set,

- $\gamma \in \Gamma_k^I = \mathrm{Map}(I, \Gamma^k)$, and
- $f \in \mathcal{O}(\check{G}^\Delta \backslash \check{G}^I / \check{G}^\Delta)$.

The map in (ii) is characterized by the values

$$\nu(S_{I,\gamma,f}) = f(\rho_\nu(\gamma_1), \ldots, \rho_\nu(\gamma_{|I|})) \in \overline{\mathbb{Q}}_\ell$$

for $I, \gamma \in \check{G}^I$, and $f \in \mathcal{O}(\check{G}^\Delta \backslash \check{G}^I / \check{G}^\Delta)$

Recall that by the work of Peter-Weyl, we have the decomposition

$$\mathcal{O}(\check{G}^I) = \bigoplus_{W \in \mathrm{Irr}\, \check{G}^I} W^* \otimes W.$$

Hence each element $f \in \mathcal{O}(\check{G}^I)$ is written as $f = \sum_W f_W$, where each f_W is a linear combination of matrix coefficients of W, i.e.,

$$f_W(g) = \langle w^*, gw \rangle$$

for some $w \in W$ and $w^* \in W^*$.

Note that f_W is in $\mathcal{O}(\check{G}^\Delta \backslash \check{G}^I / \check{G}^\Delta)$ if and only if w and w^* are fixed by $\check{G}^\Delta$.

6. Getting Shimura varieties

6.1. **Observation.** In this subsection, we observe the condition under which we can get a Shimura variety of a given group G.

Interlude (Number field case). Given G, consider a Shimura variety Sh_G. Then $G(\mathbb{A}) \times \Gamma_{\mathbb{Q}}$ acts on the cohomology group

$$H^*_{\mathrm{cusp}}(\mathrm{Sh}_{G^*}(\overline{\mathbb{Q}}), \overline{\mathbb{Q}}_\ell) = \bigoplus_{\Pi:\ \text{cuspidal}} \Pi \otimes \rho_\Pi,$$

where ρ_Π are Galois representations. But not every G has a Shimura variety.

Shimura data of G is indexed by the following equivalent objects.

- minuscule coweight of G,
- minuscule weight of $\check{G}$, and
- minuscule irreducible representation (R, V) of $\check{G}$.

For given minuscule irreducible representation (R, V), the ρ_Π that one gets is of the form

$$\rho_\Pi \colon \Gamma_{\mathbb{Q}} \xrightarrow{\phi_\Pi} \check{G}(\overline{\mathbb{Q}}_\ell) \xrightarrow{R} \mathrm{GL}(V).$$

To produce a homomorphism $\rho \colon \Gamma \longrightarrow \check{G}$, one can appeal to *Tannaka duality*.

Consider the category $(\mathrm{Rep}(\check{G}))$ which has a tensor product. For the forgetting functor

$$F\colon\ \mathrm{Rep}(\check{G}) \longrightarrow (\mathrm{Vect}),$$

we have

$$\check{G} = \mathrm{Aut}(F).$$

So to define a homomorphism $\Gamma \longrightarrow \check{G}$ is equivalent to giving $\rho_V : \Gamma \longrightarrow \mathrm{GL}(V)$ for $V \in (\mathrm{Rep}(\check{G}))$ satisfying the following two conditions.

- For any G-equivariant map $T\colon V \longrightarrow W$ and for all $\gamma \in \Gamma$, we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle \rho_V(\gamma)}\downarrow & & \downarrow{\scriptstyle \rho_W(\gamma)} \\ V & \xrightarrow{T} & W. \end{array}$$

- $\rho_{V\otimes W} = \rho_V \otimes \rho_W$.

Consequently, to get a global L-parameter

$$\rho\colon \Gamma_{\mathbb{Q}} \longrightarrow \check{G},$$

we need a Shimura variety $\mathrm{Sh}_{\mathcal{O},V}$ for every representation V of $\check{G}$.

Miracle (function field case). Over function fields, one has a "Shimura variety" for every V.

6.2. **Analogues of Shimura varieties, Moduli spaces of shtukas.** In this subsection, we introduce the analogous object over function field to Shimura variety over a number field, and we get a mechanism which produce cusp forms by using "Shimura variety" at the last of this lecture note.

The following are given:

- a curve X over $\mathbb{F}_q$,
- finite subscheme $N \subset X$,
- a finite set I, and
- a scheme S over $\mathbb{F}_q$.

Definition 6.1. A *shtuka* on $X \times_{\mathbb{F}_q} S$ is the data

- a G-torsor $\mathcal{G}$ on $X \times_{\mathbb{F}_q} S$,
- a collection of points $x_i \in (X \setminus N)(S)$ for $i \in I$,
- a modification of $\mathcal{G}$, i.e., an isomorphism

$$\phi\colon \mathcal{G}|_{(X\times S)\setminus\bigcup_{i\in I}\Gamma_{x_i}} \longrightarrow {}^{\tau}\mathcal{G}|_{(X\times S)\setminus\bigcup_{i\in I}\Gamma_{x_i}},$$

  where ${}^{\tau}\mathcal{G} = (\mathrm{id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})$, and
- a trivialization of $(\mathcal{G}, \phi)$ at N.

Hence ${}^{\tau}\mathcal{G}$ is just $\mathcal{G}$ modified at the finitely many points $\{x_i\}$. Call $\{x_i\}_{i\in I}$ the *legs/paws* of shtuka.

Remark 6.2. If $I = \emptyset$, a shtuka of $X \times S$ is just a G-torsor on X over $\mathbb{F}_q$.

The moduli space of G-torsors on X with trivialization is denoted by $\mathrm{Bun}(G, N)$ and Weil showed the equality

$$\mathrm{Bun}(G,N)(\mathbb{F}_q) = G(k)\backslash G(\mathbb{A})/K_N.$$

For any irreducible representation $W = \boxtimes_{i\in I} W_i$ of $\check{G}^I$, there is a notion of "a modification bounded by W_i at x_i for each i".

Definition 6.3. Let $\mathrm{Cht}_{N,I,W}$ be the (reduced) moduli stack of shtukas with $|I|$ legs/ paws and modification bounded by $W \in \mathrm{Irr}\, \check{G}^I$. Then we have

$$P_{N,I,W} \colon \mathrm{Cht}_{N,I,W} \longrightarrow (X \setminus N)^I$$

taking legs.

Remark 6.4. If $I = \emptyset$ or if $W = 1$, then $\mathrm{Cht}_{N,I,W}$ is just the constant stack $\mathrm{Bun}_G(\mathbb{F}_q)$ over $X \setminus N^I$.

Definition 6.5. We denote

$$\mathcal{H}_{N,I,W} = R^0(P_{N,I,W})_!(\mathrm{IC}_{\mathrm{Cht}_{N,I,W}}),$$

where IC is the intersection cohomology. This is equipped with the action of Hecke operator $\mathcal{H}_N$ via Hecke correspondences. This is also equipped with the action of "partial Frobenius" which is eventually these action of Γ_k^I at the generic point of $X \times N$.

Let us list the functional properties of $\mathcal{H}_{N,I,W}(\curvearrowleft \Gamma_k^I \times \mathcal{H}_N)$.

(a) For fixed I, the correspondence $W \mapsto \mathcal{H}_{N,I,W}$ gives a $\overline{\mathbb{Q}}_\ell$-linear functor

$$(\mathrm{Rep}\, \check{G}) \longrightarrow (\mathrm{Rep}(\Gamma_k^I \times \mathcal{H}_N)).$$

(b) Any map $\zeta \colon I \longrightarrow J$ induces isomorphism

$$\chi_\zeta \colon \mathcal{H}_{I,W} \longrightarrow \mathcal{H}_{J,I^*(W)}$$

which is functorial in W.

(c) We have the equality

$$\mathcal{H}_{\emptyset,1} = C_c(G(k)\backslash G(\mathbb{A})/K_N)$$

and $\mathcal{H}_{\emptyset,1}$ contains $\mathcal{A}_{\mathrm{cusp},N}$

Consequently, for given data $W \in \mathrm{Rep}(\check{G}^I), w \in W^{\check{G}^\Delta}, w^* \in (W^*)^{\check{G}^\Delta}$, and $\gamma \in \Gamma_k^I$, we have the following commutative diagram like the kids' dance song "Hokey Pokey" (search the video).

$$\begin{array}{cccccccc}
\text{In} & \mathcal{H}_{\{0\},W} & \xrightarrow{\chi_\zeta^{-1}} & \mathcal{H}_{I,W} & \xrightarrow[\text{“}HokeyPokey\text{''}]{\gamma} & \mathcal{H}_{I,W} & \xrightarrow{\chi_\zeta} & \mathcal{H}_{\{0\},W} \\
& {\scriptstyle w}\uparrow & & & & & & \downarrow{\scriptstyle w^*} \\
\text{Out} & \mathcal{A}_{\mathrm{cusp},N} & & & \xrightarrow[\mathcal{H}_N\text{-equiv.}]{S_{I,\gamma,W,w,w^*}} & & & \mathcal{A}_{\mathrm{cusp},N}.
\end{array}$$

7. Applications of the globalization theorem

7.1. **Langlands-Shahidi theory.** The main application of the globalization theorem is Langlands-Shahidi theory in positive characteristic, which is a recent result of Luis Lomelí. Here we give a sketch of this theory.

Let G be a connected reductive group over a local field F of positive characteristic. Let P be a parabolic subgroup of G, H a Levi component and N the unipotent radical. Passing to the dual group $\check{G}$, there is a corresponding parabolic subgroup $\check{P} = \check{H}\check{N}$. Then $\check{H}$ acts on $\mathrm{Lie}(\check{N})$ by adjoint action. We denote the irreducible decomposition of this action by

$$(\mathrm{Lie}(\check{N}), \mathrm{Ad}) = \bigoplus_i (r_i, V_i).$$

Take an irreducible admissible representation π of H. Assuming the local Langlands correspondence, π is associated with an L-parameter

$$\phi_\pi \colon WD_F \longrightarrow \check{H}.$$

Due to Deligne, Langlands (characteristic 0) and Laumon (positive characteristic), we have the local L-factor $L(s, r_i \circ \phi_\pi)$ and the local epsilon factor $\varepsilon(s, r_i \circ \phi_\pi, \psi)$. Langlands-Shahidi theory aims to define these local factors directly from π, without invoking local Langlands correspondence, i.e., we want to define $L(s, \pi, r_i)$ and $\varepsilon(s, \pi, r_i, \psi)$. As usual, it is sufficient to define a coarser invariant $\gamma(s, \pi, r_i, \psi)$, a gamma factor, which is related to these two factors by

$$\gamma(s, \pi, r_i, \psi) = \varepsilon(s, \pi, r_i, \psi) \frac{L(1-s, \pi^\vee, r_i)}{L(s, \pi, r_i)}.$$

Example 7.1. (1) Let $G = \mathrm{GL}_{m+n}$ and $P = H \ltimes N$ be a parabolic subgroup of G with Levi component $H = \mathrm{GL}_m \times \mathrm{GL}_n$. Then the dual groups are $\check{G} = \mathrm{GL}_{m+n}(\mathbb{C})$ and

$$\check{P} = (\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \ltimes \check{N}.$$

The decomposition of $\mathrm{Lie}(\check{N})$ is given by

$$\mathrm{Lie}(\check{N}) = (\mathrm{Std}_m)^\vee \boxtimes (\mathrm{Std}_n)^\vee,$$

where Std_k denotes the standard representation of GL_k. Given an irreducible admissible representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{GL}_m \times \mathrm{GL}_n$, we can define

$$\gamma(s, \pi_1 \boxtimes \pi_2, \mathrm{Std}_m \boxtimes \mathrm{Std}_n),$$

which is usually written simply by $\gamma(s, \pi_1 \times \pi_2, \psi)$ and called the *Rankin-Selberg gamma factor.*

(2) Let $G = \mathrm{Sp}_{2n}$ and $P = H \ltimes N$ be a parabolic subgroup of G with Levi component $H = \mathrm{GL}_a \times \mathrm{Sp}_{2n-2a}$. Then we have $\check{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$ and

$$\check{P} = (\mathrm{GL}_a(\mathbb{C}) \times \mathrm{SO}_{2n-2a+1}(\mathbb{C})) \ltimes N^\vee.$$

The decomposition of $\mathrm{Lie}(\check{N})$ is given by

$$\mathrm{Lie}(\check{N}) = r_1 \oplus r_2,$$

where $r_1 = \mathrm{Std}_a \boxtimes \mathrm{Std}_{\mathrm{SO}_{2n-2a+1}}$ and $r_2 = \wedge^2 \mathrm{Std}_a$. Given an irreducible admissible representation $\pi \boxtimes \sigma$ of $\mathrm{GL}_a \times \mathrm{Sp}_{2n-2a}$, we get two gamma factors; one is Rankin-Selberg type $\gamma(s, \pi \boxtimes \sigma, r_1, \psi)$, which is usually denoted by $\gamma(s, \pi \times \sigma, \psi)$ and the other is the *exterior square gamma factor*

$$\gamma(s, \pi \boxtimes \sigma, r_2, \psi) = \gamma(s, \pi, \wedge^2, \psi).$$

Theorem 7.2 (Shahidi for number field case, Lomelí for function field case)**.** Notation are as above.

(1) For a generic irreducible admissible representation π of H, one can define

$$\gamma(s, \pi, r_i, \psi) \in \mathbb{C}(q_F^{-s})$$

satisfying the following properties:

- If $H = T$ is a torus, then $\gamma(s, \pi, r_i, \psi)$ is compatible with local Langlands correspondence for tori.
- (Multiplicativity) If π is a subrepresentation of $\mathrm{Ind}_Q^H(\tau)$ for some parabolic subgroup Q of H and its generic irreducible admissible representation τ, then $\gamma(s, \pi, r_i, \psi)$ can be expressed by the Langlands-Shahidi gamma factor for τ.
- (Global functional equation) Fix a global additive character $\Psi = \prod_v \Psi_v$ on $k\backslash\mathbb{A}$. If Π is a globally generic cuspidal automorphic representation of $H(\mathbb{A})$, one has

$$L^S(s, \Pi, r_i) = \prod_{v \in S} \gamma(s, \Pi_v, r_i, \Psi_v) L^S(1-s, \Pi^\vee, r_i),$$

for some suitable finite set S of places of k.

Moreover, these properties characterize $\gamma(s, \pi, r_i, \psi)$.

(2) The gamma factors $\gamma(s, \pi, r_i, \psi)$ satisfy the following additional property:

- (Local functional equation)

$$\gamma(s, \pi, r_i, \psi)\gamma(1-s, \pi^\vee, r_i, \overline{\psi}) = 1.$$

The strategy to prove this theorem is as follows: by the multiplicativity property, we can reduce the problem to the supercuspidal case. Given a supercuspidal representation, we globalize it and use global functional equation.

In the number field case, one can use "shrinking support" method to globalize supercuspidal representations, sacrificing one archimedean place where we already know every representation is obtained from principal series representations. On the other hand, in the function field case, we cannot use the same method since there is no such a favorable place we can sacrifice.

7.2. **Stability of the Langlands-Shahidi gamma factors.**

Theorem 7.3. If π_1 and π_2 are generic admissible representations of H with the same central character, then we have

$$\gamma(s, \pi_1 \otimes \chi, r_i, \psi) = \gamma(s, \pi_2 \otimes \chi, r_i, \psi).$$

for any sufficiently ramified characters χ of H.

Let us explain why we expect this property for gamma factors. Assuming local Langlands correspondence, π is associated with an L-parameter

$$\phi_\pi \colon WD_F \longrightarrow {}^L H.$$

Suppose $\gamma(s, \pi, r_i, \psi) = \gamma(s, r_i \circ \phi_\pi, \psi)$, which is a part of the conjectural naturality of local Langlands correspondence. In Galois theoretic context, the stability of gamma factors holds by Deligne and Henniart. Our theorem will be shown by passing to the Galois side and applying this result.

Recall the result of V. Lafforgue. Let k be a global field of characteristic $p > 0$. V. Lafforgue constructed a map for

$$\left\{ \begin{array}{c} \text{cuspidal automorphic} \\ \text{representations of } H(\mathbb{A}) \end{array} \right\} \xrightarrow{\text{(VL)}} \left\{ \begin{array}{c} \rho_\ell \colon \operatorname{Gal}(k^{sep}/k) \longrightarrow {}^L H(\overline{\mathbb{Q}}_\ell) \\ \text{continuous } \ell\text{-adic Galois representations} \end{array} \right\},$$

where l is a prime number different from p.

If we fix an isomorphism $\iota_\ell \colon \widehat{\overline{\mathbb{Q}}}_\ell \xrightarrow{\cong} \mathbb{C}$, the local component $\rho_{\ell,v}$ of ρ_ℓ can be expressed by

$$\phi_v \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L H(\mathbb{C}).$$

Note that this may not be Frobenius semi-simple and let us denote its Frobenius semi-simplification by ϕ_v^{Frob}

This map satisfies the following properties: Let π be an cuspidal automorphic representation of $H(\mathbb{A})$.

(a) For almost all v at which π_v is unramified, $\rho_{\ell,v}$ gives the L-parameter of π_v

(b) Let $Z = Z(H_F)^\circ$ be the connected center of H_F. We denote by ρ_Z the projection map

$$^L H_F \longrightarrow {}^L H_F/({}^L H_F^\circ)^{\mathrm{der}} = {}^L Z.$$

Then the central character ω_π of π corresponds to $\rho_Z \circ \rho_{\pi,\ell}$.

Lemma 7.4. For an irreducible generic admissible representation π of H_F, there is an L-parameter

$$\phi \colon WD_F \longrightarrow {}^L H_F$$

such that

- the central character ω_π corresponds to $\rho_Z \circ \phi$ under the local Langlands correspondence for tori
- $\gamma(s, \pi \otimes \chi_F, r_i, \psi) = \gamma(s, r_i \circ (\phi \otimes \chi_F), \psi)$ holds for all unitary characters χ_F of H_F.

First we deduce our theorem from this lemma. Suppose π_1 and π_2 are irreducible generic representations of H_F with the same central character. Assuming the above lemma, there are L-parameters ϕ_i corresponding to π_i. By the first condition in the lemma, $\rho_Z \circ \phi_1 = \rho_Z \circ \phi_2$. For any finite dimensional representation R of ${}^L H_F$, $\det R$ factors through

$$\rho_Z \colon {}^L H_F \longrightarrow {}^L H_F / (\check{H}_F)^{\mathrm{der}}.$$

Hence we get $\det R \circ \phi_1 = \det R \circ \phi_2$. Then Theorem 7.3 follows from the result of Deligne and Henniart.

Proof of lemma. We may assume that π is a supercuspidal representation. Applying the globalizing theorem (or rather its corollary), we get a global field k and a generic cuspidal automorphic representation Π satisfying $k_{v_0} \cong F$, $\Pi_{v_0} \cong \pi$, and Π_v is contained in some principal series representation for any $v \neq v_0$. Fix a non-trivial additive character $\Psi = \prod_v \Psi_v$ of $k \backslash \mathbb{A}$.

Applying the globalizing theorem again to χ_F, we obtain an automorphic character χ such that $\chi_{v_0} = \chi_F$.

By the result of V. Lafforgue, Π corresponds to a Galois representation ρ_Π. Let S be a large large finite set of places which does not contain v_0, such that for any $v \notin S$, Π_v and χ_v are unramified and $\rho_{\Pi,v}$ gives the L-parameter of Π_v.

By the global functional equation obtained from Langlands-Shahidi theory,

$$L^S(s, \Pi \otimes \chi, R) = \prod_{v \in S \cup \{v_0\}} \gamma(s, \Pi_v \otimes \chi_v, R, \Psi_v) L^S(1-s, \Pi^\vee \otimes \chi^{-1}, R).$$

On the other hands, the functional equation of L-functions for Galois representations leads

$$L^S(R \circ (\rho_\Pi \otimes \chi)) = \prod_{v \in S \cup \{v_0\}} \gamma(s, R \circ \rho_{\Pi,v} \otimes \chi_v, \Psi_v) L^S(1-s, R \circ (\rho_\Pi^\vee \otimes \chi^{-1})).$$

Note that we have

$$L^S(s, \Pi \otimes \chi, R) = L^S(R \circ (\rho_\Pi \otimes \chi_F))$$

and

$$L^S(1-s, \Pi^\vee \otimes \chi^{-1}, R) = L^S(1-s, R \circ (\rho_\Pi^\vee \otimes \chi^{-1})).$$

Hence we get

$$\prod_{v \in S \cup \{v_0\}} \gamma(s, \Pi_v \otimes \chi_v, R, \Psi_v) = \prod_{v \in S \cup \{v_0\}} \gamma(s, R(\circ \rho_{\Pi,v} \otimes \chi_v), \Psi_v)$$

Since Π_v for $v \in S$ is contained in some principal series representation, we get

$$\gamma(s, \Pi_v \otimes \chi_v, R, \Psi_v) = \gamma(s, R \circ (\phi_{\Pi,v} \otimes \chi_v), \Psi_v)$$

from the multiplicativity of the gamma factors and the compatibility of local Langlands correspondence for tori. Therefore we get

$$\gamma(s, \Pi_{v_0} \otimes \chi_{v_0}, R, \Psi_{v_0}) = \gamma(s, R \circ (\rho_{\Pi, v_0} \otimes \chi_{v_0}), \Psi_{v_0}) \prod_{v \in S} \frac{\gamma(s, R \circ (\rho_{\Pi,v} \otimes \chi_v), \Psi_v)}{\gamma(s, R \circ (\phi_{\Pi,v} \otimes \chi_v), \Psi_v)}.$$

We may assume that χ_v for $v \in S$ is sufficiently ramified, and if this is the case,

$$\prod_{v \in S} \frac{\gamma(s, R \circ (\rho_{\Pi,v} \otimes \chi_v), \Psi_v)}{\gamma(s, R \circ (\phi_{\Pi,v} \otimes \chi_v), \Psi_v)} = 1$$

by the result of Deligne and Henniart. This implies the lemma. □

The following corollary is a consequence of the proof of the above lemma.

Corollary 7.5. Let Π be a generic cuspidal automorphic representation of $H(\mathbb{A})$ and X be a Hecke character. Fix a non-trivial additive character Ψ of $k \backslash \mathbb{A}$. Then for any place v, we have

$$\gamma(s, \Pi_v \otimes \chi_v, R, \Psi_v) = \gamma(s, R \circ (\rho_{\Pi,v} \otimes \chi_v), \Psi_v).$$

8. Construction of local Langlands correspondence

8.1. **Plancherel measures.** Let

- F a local field of characteristic p,
- $G_F \supset P_F = H_F N_F$ be a parabolic subgroup of G_F with Levi decomposition,
- π_F an irreducible representation of H_F,
- χ_F a character of H_F, and
- ψ a non-trivial additive character of F.

Then, we have standard intertwining operators [9]:

$$\mathrm{Ind}_{P_F}^{G_F}(\pi_F \otimes \chi_F) \xrightarrow{M_{P_F, \overline{P}_F, \psi}(\pi_F \otimes \chi_F)} \mathrm{Ind}_{\overline{P}_F}^{G_F}(\pi_F \otimes \chi_F) \xrightarrow{M_{\overline{P}_F, P_F, \psi}(\pi_F \otimes \chi_F)} \mathrm{Ind}_{P_F}^{G_F}(\pi_F \otimes \chi_F).$$

Here, $\overline{P}_F$ is the opposite of P_F.

Since these induced representations are irreducible for general χ_F, composition of two intertwining operators above are actually a scalar.

Definition 8.1. We define a $\mathbb{C}$-valued meromorphic function μ in χ_F by

$$\mu(\pi_F \otimes \chi_F, \psi)^{-1} = M_{\overline{P_F}, P_F, \psi}(\pi_F \otimes \chi_F) \circ M_{P_F, \overline{P_F}, \psi}(\pi_F \otimes \chi_F).$$

This function is called the *Plancherel measure*. Note that μ depends on the choice of Haar measures. For a precise normalization, see [3]. Its poles tell us something about reducibility points of induced representations as we will see later. One of the achievement of Langlands-Shahidi theory is breaking up this invariant into smaller pieces;

Proposition 8.2. If π is generic, we have

$$\mu(\pi_F, \psi) = \prod_i \gamma^{\mathrm{LS}}(\pi_F, r_i, \psi)\gamma^{\mathrm{LS}}(\pi_F^\vee, r_i, \overline{\psi}),$$

where $\mathrm{Lie}(\check{N}_F) = \bigoplus r_i$ as a representation of $\check{H}_F$. Here, the superscript LS means that the gamma factor is the one obtained from the Langlands-Shahidi method.

Similarly as in the last section, we have the following equation between representation theoretic invariant and Galois theoretic invariant.

Lemma 8.3. If π_F is a supercuspidal representation of H_F, then there is an L-parameter

$$\phi_F \colon WD_F \longrightarrow {}^L H_F$$

such that

$$\mu(\pi_F, \psi) = \prod_i \gamma^{\mathrm{Gal}}(r_i \circ \phi_F, \psi_v)\gamma^{\mathrm{Gal}}(r_i^\vee \circ \phi_F, \overline{\psi}_v).$$

Proposition 8.4.
(1) Given irreducible representations π_1, π_2 of H_F with the same central character, we have

$$\mu(\pi_1 \otimes \chi_F, \psi) = \mu(\pi_2 \otimes \chi_F, \psi)$$

for every sufficiently ramified character χ_F.

(2) If $\Pi = \bigotimes_v' \Pi_v$ is a cuspidal automorphic representation with $\rho_{\Pi,\ell}$ via a map (VL), we have

$$\mu(\Pi_v, \Psi_v) = \prod_i (\gamma^{\mathrm{Gal}}(r_i \circ \rho_{\Pi,\ell,v}, \Psi_v)\gamma^{\mathrm{Gal}}(r_i^\vee \circ \rho_{\Pi,\ell,v}, \overline{\Psi}_v))$$

for every place v.

Proof. The proof of (1) is analogous to that of Theorem 7.3. We only give the sketch. By multiplicativity of gamma factors, we can reduce to the supercuspidal case. Hence we may assume that π_i are supercuspidal. Admitting Lemma 8.3, the assertion follows from the stability of gamma factors by Deligne and Henniart. This completes the proof. □

Our goal is to construct a Local Langlands Correspondence

$$\mathcal{L} \colon \mathrm{Irr}\, H_F \longrightarrow \{\ L\text{-parameters for } H_F\}$$

via V. Lafforgue's Global Langlands Correspondence.

Consider the case that H_F is classical groups. Let $E = F$, or a quadratic extension of F. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional vector space V with ε-hermitian form $\langle\cdot,\cdot\rangle$. Define $H_F = \mathrm{Aut}(V, \langle\cdot,\cdot\rangle)^\circ$. Then,

For these H_F, L-parameters are simply equivalent classes of conjugate self-dual representations $\rho \colon WD_E \to \mathrm{GL}_N(\mathbb{C})$ with some sign $\varepsilon(H_F)$.

H_F	Sp_{2n}	SO_{2n}	SO_{2n+1}	U_n
$\check{H}_F$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C})$
${}^L H_F$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{O}_{2n}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$

In [1], [4], using Langlands Shahidi-theory and the converse theorem, one obtains a functorial lifting

$$\left\{ \begin{array}{c} \text{generic cuspidal automorphic} \\ \text{representations of } H \text{ over } k \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{automorphic representations} \\ \text{of } \mathrm{GL}_N \end{array} \right\}.$$

From this, one gets a local Langlands correspondence $\mathcal{L}^{\mathrm{LS}}$ for supercuspidal representations.

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{gen,sc}}\, H_F & \longrightarrow & \mathrm{Irr}_{\mathrm{sc}}\, \mathrm{GL}_N(F) \\ {\scriptstyle \mathcal{L}^{\mathrm{LS}}} \downarrow & \searrow & \updownarrow {\scriptstyle \text{LLC for } \mathrm{GL}_N} \\ \{\, \phi \colon W_E \to {}^L H_F \,\} & \hookrightarrow & \{\, \psi \colon WD_E \to \mathrm{GL}_N(\mathbb{C}) \,\} \end{array}$$

Here, $\mathrm{Irr}_{\mathrm{gen,sc}}\, H_F$ is the set of the equivalent classes of supercuspidal representations of H_F and $\mathrm{Irr}_{\mathrm{sc}}\, \mathrm{GL}_N(F)$ is the set of the equivalent classes of supercuspidal representations of $\mathrm{GL}_N(F)$.

Here $\mathcal{L}$ is characterized by the equality

$$\gamma^{\mathrm{LS}}(\pi \times \tau, \psi) = \gamma^{\mathrm{Gal}}(\phi_\pi \otimes \phi_\tau, \psi)$$

for every irreducible representations τ of $\mathrm{GL}_N(F)$. The right hand side is the gamma factor of L-parameters.

We assume the *Working Hypothesis*: Suppose $H_F = \mathrm{SO}$ or Sp. We assume that the image of $\mathcal{L}^{\mathrm{LS}}$ contains a tamely ramified almost irreducible parameter ϕ_1. We denote by π_1 the irreducible generic supercuspidal representation of H_F corresponding to ϕ_1.

Remark 8.5. (1) If the local descent is extended to function fields, then one would know $\mathcal{L}^{\mathrm{LS}}$ is surjective.

(2) In positive characteristic, to this ϕ, Debacker-Reeder associates an L-packet of depth 0 supercuspidal representations. Savin checked that for a generic supercuspidal representation π in this packet, we have

$$\gamma^{\mathrm{LS}}(\pi \times \tau, \psi) = \gamma^{\mathrm{Gal}}(\phi \otimes \phi_\tau, \psi).$$

Let $\mathrm{Irr}\, H_F$ denote the set of equivalent classes of irreducible admissible representations of H_F.

Theorem 8.6. Suppose that F is a function field with positive characteristic $p \neq 2$ and H_F is quasi-split. Assume the Working Hypothesis, then there exists a map

$$\mathcal{L} \colon \mathrm{Irr}\, H_F \to \{L\text{-parameters for } H_F\}$$

such that for $\pi \in \operatorname{Irr} H_F$ and $\phi_\pi = \mathcal{L}(\pi)$, the following properties hold:

- π is a discrete series representation if and only if ϕ_π is elliptic,
- π is temped if and only if $\phi|_{W_E}$ is bounded,
- if π is generic, then $\gamma^{\mathrm{LS}}(\pi \times \tau, \psi) = \gamma^{\mathrm{Gal}}(\phi_\pi \otimes \phi_\tau, \psi)$ i.e., $\phi_\pi = \mathcal{L}^{\mathrm{LS}}(\pi)$,
- for general π, $\mu(\pi \times \tau, \psi) = \mu^{\mathrm{Gal}}(\pi \otimes \tau, \psi)$, and
- $\mathcal{L}$ is compatible with Langlands classification.

Moreover, $\mathcal{L}$ is characterized by these properties.

We then ask the following questions.

(a) Is $\mathcal{L}$ surjective?
(b) Are the fibers finite?
(c) Is there a refined classification of fibers?

In order to consider the above questions, we will construct $\mathcal{L}$ for supercuspidal representations π of H_F. By the globalization theorem, we can construct the following items:

- a global field k with $k_{v_0} \cong F$,
- a quasi-split group H over k with $H_{v_0} \cong H_F$, and
- a cuspidal representation Π of $H(\mathbb{A})$ such that

$$\begin{cases} \Pi_{v_0} \cong \pi_F. \\ \Pi_{v_1} \cong \pi_1 \text{ as in working hypothesis.} \\ \Pi_v \text{ is contained in a principal series, for every } v \neq v_0, v_1. \end{cases}$$

Take an automorphic representation Π. Let ρ_Π be the L-parameter of Π via a map (VL).

Proposition 8.7. (1) At v_1, we have $\rho_{\Pi,\ell,v} = \phi_1$.
(2) ρ_Π is pure of weight 0.
(3) At v_0, ρ_{Π,ℓ,v_0} is an elliptic L-parameter of H_F.

Proof. (1) By the result on Plancherel measure, we have

$$\begin{aligned} \mu(s, \pi_1 \times \tau, \psi) \quad &= \gamma(s, \rho^\vee_{\Pi,\ell,v_1} \otimes \phi_\tau, \psi)\gamma(-s, \rho_{\Pi,\ell,v_1} \otimes \phi^\vee_\tau, \overline{\psi}) \\ &\quad \times \gamma(2s, R \circ \phi_\tau, \psi)\gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}), \end{aligned}$$

for all irreducible supercuspidal representations τ of GL_r with arbitrary r. Here, R is one of the following representation of $\mathrm{GL}_N(\mathbb{C})$ depending on H_F:

$$R = \begin{cases} \mathrm{Sym}^2 \\ \wedge^2 \\ \mathrm{Asai}^\pm \end{cases}$$

We write the right-hand side by $\mu^{\mathrm{Gal}}(\rho_{\Pi,\ell,v_1} \times \phi_\tau, \psi)$.

On the other hand, since both π_1 and τ are generic, Langlands-Shahidi theory (Proposition 8.2 and Proposition 8.4) provides us the following equation:

$$\begin{aligned}\mu(s,\pi_1\times\tau,\psi) &= \gamma^{\mathrm{LS}}(s,\pi_1\times\tau,\psi)\gamma^{\mathrm{LS}}(-s,\pi^\vee\times\tau^\vee,\overline{\psi})\\ &\qquad\times\gamma^{\mathrm{LS}}(2s,\tau,R,\psi)\gamma^{\mathrm{LS}}(-2s,\tau^\vee,R^\vee,\overline{\psi})\\ &= \mu^{\mathrm{Gal}}(\phi_1\times\phi_\tau,\psi)\end{aligned}$$

Note that the local Langlands correspondence preserves gamma factors only up to some roots of unity, but this ambiguity is canceled in the above formula.

From this, we get

$$\gamma(s,\rho^\vee_{\Pi,\ell,v_1}\otimes\phi_\tau,\psi)\gamma(-s,\rho_{\Pi,\ell,v_1}\otimes\phi^\vee_\tau,\overline{\psi})=\gamma(s,\phi^\vee_1\otimes\phi_\tau,\psi)\gamma(-s,\phi_1\otimes\phi^\vee_\tau,\overline{\psi}).$$

Since ϕ_1 is an almost irreducible Galois representation, this equation shows that $\rho_{\Pi,\ell,v_1}=\phi_1$.

(2) follows from a conjecture of Deligne in Weil II which was shown by L. Lafforgue. Since $\rho_{\Pi,\ell}$ is "almost irreducible", it is pure of weight 0.

(3) Deligne showed in Weil —— (Theorem 1.8.4), that if ρ_{Π,ℓ,v_0} is pure of weight 0, then ρ_{Π,ℓ,v_0} is a tempered L-parameter for GL_N. Moreover, by Proposition 8.4, one has

$$\begin{aligned}\mu(s,\pi\times\tau,\psi)=\gamma(s,\rho^\vee_{\Pi,\ell,v_0}\otimes\phi_\tau,\psi)\gamma(-s,\rho_{\Pi,\ell,v_0}\otimes\phi^\vee_{v_0},\overline{\psi})\\ \times\gamma(2s,R\circ\phi_\tau,\overline{\psi})\gamma(-2s,R^\vee\circ\phi_\tau,\overline{\psi})\end{aligned}$$

for any ϕ_τ. Now, if ψ_τ is not conjugate-self-dual, then it follows by [9] that the left hand side is non-zero and hence so is right hand side. This implies ρ_{Π,ℓ,v_0} does not contain any non-conjugate-self-dual summand. Further, it follows by [9] that the left hand side has a zero of order at most 2, which implies that ρ_{Π,ℓ,v_0} has multiplicity free. Hence, ρ_{Π,ℓ,v_0} is a discrete parameter of H_F. □

Next, we will show that ρ_{Π,ℓ,v_0} in the previous proposition only depends on $\Pi_{v_0}\cong\pi_F$. In other words, it is independent of several choice of globalization.

Proposition 8.8 (Independence)**.** Suppose

- k and k' are global fields with $k_{v_0}\cong k_{v'_0}\cong F$.
- H and H' are reductive groups over k and k' such that $H_{v_0}\cong H_{v'_0}\cong H_F$, respectively.
- Π and Π' are cuspidal representations of H and H', respectively.
- $\Pi_{v_0}\cong\Pi_{v'_0}\cong\pi_F$.
- $\rho_{\Pi,\ell}$ and $\rho_{\Pi,\ell'}$ are pure of weight 0, where ℓ,ℓ' are two primes different from the characteristic of F.

Then, $\rho_{\Pi,v_0,\ell}$ and $\rho_{\Pi',v'_0,\ell'}$ give the same L-parameter of H_F.

8.2. **Reducibility of generalized principal series.** Let τ be an irreducible representation of GL_r. We define the induced representation $I(s,\pi\times\tau)$ by

$$I(s,\pi\times\tau)=\mathrm{Ind}_P^G(\pi\boxtimes\tau|\mathrm{det}|^s),\qquad s\in\mathbb{C}.$$

We call $I(s, \pi \times \tau)$ a *principal series*. Define $c \in \mathrm{Gal}(E/F)$ by $\langle c \rangle = \mathrm{Gal}(E/F)$.

Lemma 8.9 (Harish-Chandra, Silberger [7], [8])**.** (a) When $\tau^c = \tau^\vee$, then $I(s, \pi \times \tau)$ is irreducible for all $s \in \mathbb{R}$.

(b) When $\tau^c \neq \tau^\vee$, then $I(0, \pi \times \tau)$ is reducible if and only if $\mu(0, \pi \times \tau) \neq 0$. In that case, $I(s, \pi \times \tau)$ is irreducible for every $s \neq 0$ and $\mu(s, \pi \times \tau)$ is holomorphic for every $s \in \mathbb{R}$.

(c) When $\tau^c \neq \tau^\vee$ and $\mu(0, \pi \times \tau) = 0$, then $I(s_0, \pi \times \tau)$ is reducible if and only if $\mu(s_0, \pi \times \tau) = \infty$ for $s_0 > 0$. In that case, such a s_0 is unique and the pole of μ at s_0 is simple and zero of μ at $s = 0$ is of order 2 and μ has no zeros elsewhere.

Definition 8.10. For an L-parameter ϕ_π, let

$$\rho = \bigoplus_i \rho \otimes S_{a_i},$$

where S_a is the irreducible a-dimensional representation of $SL_2(\mathbb{C})$. Set

$$\mathrm{Jord}_\rho(\phi_\pi) = \{a \in \mathbb{Z}_{\geq 0} \mid \rho \otimes S_a \subset \phi_\pi\}$$

for each ρ. Then, we say ϕ_π is *sans trou* if, for all ρ such that $\mathrm{Jord}(\rho)$ is non-empty and $2 < a \in \mathrm{Jord}_\rho(\phi_\pi)$, then $a - 2 \in \mathrm{Jord}_\rho(\phi_\pi)$.

Proposition 8.11. If π is supercuspidal and $\phi_\pi = \mathcal{L}(\pi)$, then ϕ_π is sans trou.

Corollary 8.12. $I(s_0, \tau \otimes \pi)$ is reducible if and only if $\tau^\vee = \tau^c$ and one of the following conditions hold

(1) $s_0 = (a_\tau(\pi) + 1) \geq 1$ with $a_\tau(\pi) = \max\ \mathrm{Jord}_{\phi_\tau}(\phi_\pi)$.
(2) s_0 if and only if $\mathrm{Jord}_{\phi_\tau}(\phi_\pi)$ is empty and $\varepsilon(\phi_\pi) = -\varepsilon(H_F)$.
(3) $s_0 = 0$ if and only if $\mathrm{Jord}_{\phi_\tau}(\phi_\pi)$ is empty and $\varepsilon(\phi_\pi) = \varepsilon(H_F)$.

This corollary is the assumption (BA) in Mœglin-Tadić [5].

References

[1] Cogdell, J. W., Kim, H. H., Piatetski-Shapiro, I. I., Shahidi, F., *Functoriality for the classical groups*, Publ. Math. Inst. Hautes Études Sci. No. 99 (2004), 163-233.

[2] Conrad, B., Gabber, O., Prasad, G., *Pseudo-reductive groups*, Second edition, New Mathematical Monographs, 26. Cambridge University Press, Cambridge, 2015.

[3] Gan, W. T., Ichino, A., *Formal degrees and local theta correspondence*, Invent. Math. 195 (2014), no. 3, 509-672.

[4] Lomelí, L. A., *Functoriality for the classical groups over function fields*, Int. Math. Res. Not. 2009, no. 22, 4271-4335.

[5] Mœglin, C., Tadić, M., *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc. 15 (2002), no. 3, 715-786.

[6] Oesterlé, J., *Nombres de Tamagawa et groupes unipotents en caractristique p*, Invent. Math. 78 (1984), no. 1, 13-88.

[7] Silberger, A. J., *Introduction to harmonic analysis on reductive p-adic groups* Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971-1973. Mathematical Notes, 23. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979.

[8] Silberger, A. J., *Special representations of reductive p-adic groups are not integrable*, Ann. of Math. (2) 111 (1980), no. 3, 571-587.

[9] Waldspurger, J.-L., *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu 2 (2003), no. 2, 235-333.