GROTHENDIECK INEQUALITY, RANDOM MATRICES AND QUANTUM EXPANDERS

GILLES PISIER
(RECORDED BY KEI HASEGAWA)

Abstract. This note is based on SGU Special lectures by Gilles Pisier at Kyoto University from March 12 to March 18, 2015.

1. Introduction

The main topic of this note is a fundamental theorem in the metric theory of tensor products due to Grothendieck [4] in 1953. Now it is called Grothendieck's theorem (or Grothendieck's inequality), and we will refer to it as GT (or GI) for short. There are many formulations of GT, and Theorem 1.1 below is an elementary one due to Lindenstrauss and Pełczyński [15]. We refer to Pisier's paper [22] (and its "UNCUT" version, avairable at his homepage) for GT and its related topics, including connection with computer science, graph theory, and quantum mechanics (see also [11]).

In what follows, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. For any normed space X, S_X and B_X denote the unit sphere and the closed unit ball of X, respectively. Let H be the separable Hilbert space ℓ_2 over $\mathbb{K}$. For any $a = [a_{ij}] \in M_n(\mathbb{K})$, we define two norms $\|a\|_h$ and $\|a\|_\vee$ by

$$\|a\|_h = \sup\{|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle| \mid x_i, y_j \in S_H\}, \quad \|a\|_\vee = \sup\{|\sum_{i,j=1}^n a_{ij} s_i t_j| \mid s_i, t_j \in S_\mathbb{K}\}.$$

Clearly, we have $\|a\|_\vee \le \|a\|_h$.

Theorem 1.1 (GT/ inequality)**.** *There exists a constant $K > 0$ such that for any $n \in \mathbb{N}$ and any $a \in M_n(\mathbb{K})$ it follows that $\|a\|_h \le K\|a\|_\vee$.*

Note that the constant K is independent of the dimension n. The smallest K is called the *Grothendieck constant* and denoted by K_G. The exact value of K_G is still open and depends on the field $\mathbb{K}$, and hence we will write $K_G^\mathbb{R}$ and $K_G^\mathbb{C}$. It is known that $1.33 \le K_G^\mathbb{C} < K_G^\mathbb{R} < 1.782\cdots$. In Grothendieck's paper [4], it was shown that $K_G^\mathbb{R} \le \sinh(\pi/2) = 2.301\cdots$. Krivine [14] proved $K_G^\mathbb{R} \le \pi(2\log(1+\sqrt{2}))^{-1} = 1.782\cdots$ and conjectured that this is the best constant. However, in the recent paper [2], it was proved that $K_G^\mathbb{R}$ is strictly smaller than Krivine's constant.

Actually GT follows from (and equivalent to) the following:

Theorem 1.2. *There exist a constant $K > 0$, a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and (possibly non-linear) maps φ, ψ from S_H into the closed unit ball of $L_\infty(\Omega, \mathcal{A}, \mathbb{P})$ such that $\langle x, y\rangle = K\mathbb{E}[\varphi_x\psi_y]$ for $x, y \in S_H$.*

Note that $\|a\|_\vee = \sup\{|\sum a_{ij}s_i t_j| \mid s_i, t_j \in B_\mathbb{K}\}$. Thus, if φ and ψ are as in Theorem 1.2, then it follows that

$$\begin{aligned} |\sum a_{ij}\langle x_i, y_j\rangle| &= |\sum a_{ij} K\mathbb{E}[\varphi_{x_i}\psi_{y_j}]| \\ &\le K \operatorname{ess\,sup}\{|\sum a_{ij}\varphi_{x_i}(\omega)\psi_{y_j}(\omega)| \mid \omega \in \Omega\} \end{aligned}$$

$$\leq K \sup\{|\sum a_{ij} s_i t_j| \mid s_i, t_j \in B_{\mathbb{K}}\}$$
$$= K\|a\|_{\vee},$$

which implies Theorem 1.1.

Proof of Theorem 1.1 for $\mathbb{R}$ *and* $K = \sinh(\pi/2)$. Take an i.i.d. sequence $\{g_i \mid i \in \mathbb{N}\}$ of $N(0,1)$ Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\{e_j\}_{j\geq 1}$ be the standard basis of $H = \ell_2$. For any $x = \sum_j x_j e_j \in S_H$, we set $X_x := \sum_j x_j g_j \in L_2(\Omega, \mathcal{A}, \mathbb{P})$. Since $\langle x, y\rangle = \langle X_x, X_y\rangle$ holds, X_x is also $N(0,1)$ Gaussian. The key fact is that

$$\langle x, y\rangle = \sin\left(\frac{\pi}{2}\mathbb{E}[\operatorname{sgn}(X_x)\operatorname{sgn}(X_y)]\right)$$

(see [21, Lemma 5.20]). We set $F(t) := \frac{2}{\pi}\arcsin(t)$ for $t \in [-1,1]$ and $S_x := \operatorname{sgn}(X_x)$ for $x \in S_H$. Then, the above formula becomes

$$F(\langle x, y\rangle) = \mathbb{E}[S_x S_y].$$

Let $F(z)^{-1} = \sin(\frac{\pi}{2}z) = \sum_{j=0}^{\infty} a_{2j+1} z^{2j+1}$ be the Taylor expansion (i.e., $a_{2j+1} = \frac{(-1)^j}{(2j+1)!}(\frac{\pi}{2})^{2j+1}$). We then have

$$\langle x, y\rangle = F^{-1}(F(\langle x, y\rangle) = \sum_{j=0}^{\infty} a_{2j+1}\mathbb{E}[S_x S_y]^{2j+1} = \sum_{j=0}^{\infty} a_{2j+1}\mathbb{E}[S_x^{\otimes 2j+1} S_y^{\otimes 2j+1}].$$

Put $\widehat{\Omega} := \bigsqcup_{j=0}^{\infty} \Omega^{2j+1}$ and $\nu := \sum_{j=0}^{\infty} |a_{2j+1}|\mathbb{P}^{\otimes 2j+1}$ and define $\varphi, \psi : S_H \to L_\infty(\widehat{\Omega}, \nu)$ by

$$\varphi_x|_{\Omega^{2j+1}} := S_x^{\otimes 2j+1}, \quad \psi_y|_{\Omega^{2j+1}} := \operatorname{sgn}(a_{2j+1}) S_y^{\otimes 2j+1}.$$

Then, we have $\langle x, y\rangle = \langle \varphi_x, \psi_y\rangle_{L_2(\widehat{\Omega},\nu)}$. Since $\nu(\widehat{\Omega}) = \sum_j |a_{2j+1}| = \sinh(\pi/2)$, we obtain $\langle x, y\rangle = \sinh(\pi/2)\mathbb{E}[\varphi_x \psi_y]$, where the probability $\mathbb{P}$ is just ν after normalization. □

We next see that $K_G^{\mathbb{R}} \leq \frac{\pi}{2\log(1+\sqrt{2})}$ following Krivine's proof. Set $a := \log(1+\sqrt{2})$. Since $\sinh(a) = 1$ holds, $C_F := \frac{2a}{\pi}$ satisfies that $\sum_j |a_{2j+1}| C_F^{2j+1} = 1$.

Krivine's trick: There exist a Hilbert space $\mathcal{H}$ and two mappings $S_H \ni x \mapsto x' \in S_{\mathcal{H}}$ and $S_H \ni y \mapsto y'' \in S_{\mathcal{H}}$ such that $\langle x', y''\rangle_{\mathcal{H}} = F^{-1}(C_F\langle x, y\rangle_H)$.

Indeed, the Hilbert space $\mathcal{H} := \bigoplus_{j=0}^{\infty} H^{\otimes 2j+1}$ and the mappings

$$x' := \bigoplus_{j=0}^{\infty} |a_{2j+1}|^{\frac{1}{2}} C_F^{\frac{2j+1}{2}} x^{\otimes 2j+1}, \quad y'' := \bigoplus_{j=0}^{\infty} \operatorname{sgn}(a_{2j+1})|a_{2j+1}|^{\frac{1}{2}} C_F^{\frac{2j+1}{2}} y^{\otimes 2j+1}$$

satisfy the desired condition.

Proof of Theorem 1.1 for $\mathbb{R}$ *with* $K = \pi/2a$. Let $S : S_{\mathcal{H}} \to L_2(\Omega, \mathcal{A}, \mathbb{P})$ be as above. By the choice of $\mathcal{H}$, we have $F^{-1}(C_F\langle x, y\rangle_H) = \langle x', y''\rangle_{\mathcal{H}} = F^{-1}(\mathbb{E}[S_{x'} S_{y''}])$, and hence $\langle x, y\rangle = C_F^{-1}\mathbb{E}[S_{x'} S_{y''}]$. □

In the complex case, the bound $K_G^{\mathbb{C}} \leq \frac{8}{\pi(K_0+1)} = 1.4049\cdots$ is due to Haagerup [6], where K_0 is the unique solution in $(0,1)$ of the equation

$$\frac{\pi}{8}(K+1) = K\int_0^{\pi/2} \frac{\cos^2 t}{(1 - K^2\sin^2 t)^{1/2}}dt =: G(K).$$

The key of Haargerup's proof is proving that G^{-1} is analytic in $\{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < 1\}$ and its Taylor expansion $\sum_j b_{2j+1} z^{2j+1}$ satisfies that $b_1 = \pi/4$ and $b_n \leq 0$ for $n \geq 3$. Also, for standard complex Gaussian X, Y one has

$$G(\mathbb{E}[X\overline{Y}]) = \mathbb{E}[\operatorname{sgn} X \cdot \overline{\operatorname{sgn} Y}].$$

If $C_G \in [0,1]$ satisfies that $\sum_j |b_{2j+1}| C_G^{2j+1} = 1$, then the conditions on the Taylor coefficients force $C_G = \pi(K_0+1)/8$. Thus, following Krivine's argument above, we get the above estimate.

The next theorem is called the "little GT" and is an immediate consequence of GT.

Theorem 1.3 (Little GT/ inequality)**.** *There exists $k > 0$ such that for any $n \in \mathbb{N}$ and any $a = [a_{ij}] \in M_n(\mathbb{K})$ with $a \geq 0$, it follows that $\|a\|_h \leq k\|a\|_\vee$.*

The best constant k is denoted by $k_G^{\mathbb{K}}$. It is known that $k_G^{\mathbb{K}} = \|g\|_1^{-2}$, where g is a standard Gaussian random variable. Thus, we have $k_G^{\mathbb{R}} = \pi/2$ and $k_G^{\mathbb{C}} = 4/\pi$.

2. Connes–Kirchberg problem

2.1. **Connes's problem.** Let (M,τ) be a tracial probability space, i.e., M is a finite von Neumann algebra and τ is a faithful normal tracial state on M. For a given family $(M(i),\tau^i)_{i\in\mathcal{I}}$ of tracial probability spaces, we set $B := \prod_{i\in\mathcal{I}} M(i) = \{x = (x_i)_{i\in\mathcal{I}} \mid \sup_{i\in\mathcal{I}} \|x_i\|_{M(i)} < \infty\}$. Take a free ultrafilter $\mathcal{U}$ on $\mathcal{I}$ and set $I_{\mathcal{U}} := \{x \in B \mid \lim_{\mathcal{U}} \|x_i\|_{L_2(\tau^i)} = 0\}$, where $\|x_i\|_{L_2(\tau^i)} = \tau^i(x_i^* x_i)^{1/2}$. The ultraproduct $(M_{\mathcal{U}}, \tau_{\mathcal{U}})$ is the tracial probability space $M_{\mathcal{U}} = B/I_{\mathcal{U}}$ and $\tau_{\mathcal{U}}((x_i)_i) = \lim_{\mathcal{U}} \tau^i(x_i)$. The next conjecture is called *Connes's embedding conjecture* ([3]).

Conjecture 2.1 (Connes)**.** *For any tracial probability space (M,τ), there is a trace preserving embedding $(M,\tau) \subset (M_{\mathcal{U}}, \tau_{\mathcal{U}})$ with $\dim M(i) < \infty$. Equivalently, there exist $N_i \in \mathbb{N}$ and bounded linear maps $u_i : M \to M_{N_i}(\mathbb{C})$ such that $\lim_i \|u_i(a^*b) - u_i(a)^* u_i(b)\|_2 = 0$ and $\lim_i |\tau_{N_i}(u_i(a)) - \tau(a)| = 0$ for $a, b \in M$, where τ_n is the canonical normalized trace on $M_n(\mathbb{C})$.*

Theorem 2.2 (Kirchberg)**.** *There is a trace preserving embedding $(M,\tau) \subset (M_{\mathcal{U}}, \tau_{\mathcal{U}})$ with $\dim M(i) < \infty$ if and only if for any $\varepsilon > 0$, $n \in \mathbb{N}$ and unitary elements $u_1, \dots, u_n \in M$, these exist $N \in \mathbb{N}$ and unitary elements $v_1, \dots, v_n \in M_N(\mathbb{C})$ such that $|\tau(u_i^* u_j) - \tau_N(v_i^* v_j)| < \varepsilon$ for all i, j.*

2.2. **Kirchberg's problem.** A norm on a $*$-algebra A is called a C*-*norm* if it satisfies that $\|a^*a\| = \|a\|^2$, $\|a\| = \|a^*\|$, and $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$. The completion of a $*$-algebra with a C*-norm is called a C*-*algebra*. An important fact is that, *after completion*, the C*-norm is unique. It is known that every C*-algebra can be embedded into $B(H)$ for some Hilbert space H.

Let $A \subset B(H)$ and $B \subset B(K)$ be C*-algebras. Their algebraic tensor product $A \otimes_{\mathrm{alg}} B$ forms a $*$-algebra. For $t = \sum_j a_j \otimes b_j \in A \otimes_{\mathrm{alg}} B$, the *minimal* and the *maximal* tensor norms are defined by

$$\|t\|_{\min} := \|\sum_j a_j \otimes b_j\|_{B(H\otimes K)}, \quad \|t\|_{\max} := \sup \|\sum_j \pi(a_j)\sigma(b_j)\|,$$

where the supremum runs over all $*$-representations $\pi : A \to B(H')$ and $\sigma : B \to B(H')$ whose ranges commute. It is known that $\|\cdot\|_{\min}$ is the smallest C*-norm on $A \otimes_{\mathrm{alg}} B$, and hence does not depend on the embeddings $A \subset B(H)$ and $B \subset B(K)$. The completion of $A \otimes_{\mathrm{alg}} B$ by $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ are called the *minimal* and the *maximal tensor products* of A and B, and denoted by $A \otimes_{\min} B$ and $A \otimes_{\max} B$, respectively.

Definition 2.3. (A, B) is said to be a *nuclear pair* if $\|\cdot\|_{\max} = \|\cdot\|_{\min}$ on $A \otimes_{\mathrm{alg}} B$. A C*-algebra A is called *nuclear* if (A, B) is nuclear for any C*-algebra B.

Let G be a discrete group and $\mathbb{C}[G]$ be its group algebra. Any element $x \in \mathbb{C}[G]$ is of the form $\sum_{g\in G} x(g)g$ for some $x(g) \in \mathbb{C}$. The *full group* C*-*algebra* C$^*(G)$ is the completion of

$\mathbb{C}[G]$ with respect to the norm $\|x\|_{C^*(G)} := \sup \| \sum_g x(g)\pi(g)\|$, where the supremum runs over all unitary representations $\pi : G \to B(H_\pi)$.

Theorem 2.4 (Kirchberg [13]). *Let $\mathcal{C} := C^*(\mathbb{F}_\infty)$ and $\mathcal{B} := B(\ell_2)$. Then, $(\mathcal{B}, \mathcal{C})$ is a nuclear pair.*

Note that the C*-algebra $\mathcal{C}$ is "projectively" universal in the sense that every separable C*-algebra is a quotient of $\mathcal{C}$. Also, the C*-algebra $\mathcal{B}$ is "injectively" universal in the sence that every separable C*-algebra can be embedded into $\mathcal{B}$. Kirchberg showed that $\mathcal{C}$ has the *lifting property* (LP), and hence *local lifting property* (LLP). By definition, $\mathcal{B}$ has the *weak expectation property* (WEP). With these observations, in [12] Kirhberg characterized LLP and WEP in terms of tensor products with $\mathcal{C}$ and $\mathcal{B}$. Here we adopt Kirchberg's characterizations as definitions:

Definition 2.5. A C*-algebra A is said to have the WEP if $(A, \mathcal{C})$ is nuclear. We also say that A has the LLP if $(A, \mathcal{B})$ is nuclear.

One of the main results of [12] is that there is a separable non-nuclear C*-algebra A such that $(A, \bar{A})$ is nuclear, where $\bar{A} = \{\bar{a} \mid a \in A\}$ is the conjugate C*-algebra of A, i.e., $\bar{A} = A$ as an involutive ring but it has the conjugate vector space structure: $\lambda\bar{a} = \overline{\bar{\lambda}a}$ for $\lambda \in \mathbb{C}$ and $a \in A$. Kirchberg also conjectured:

Conjecture 2.6 (Kirchberg). *For $\mathcal{C} = C^*(\mathbb{F}_\infty)$ and $\mathcal{B} = B(\ell_2)$, the following are true:*

- *$(\mathcal{B}, \bar{\mathcal{B}})$ is nuclear.*
- *$(\mathcal{C}, \bar{\mathcal{C}})$ is nuclear.*

Note that $\bar{\mathcal{B}} \cong \mathcal{B}$ and $\bar{\mathcal{C}} \cong \mathcal{C}$. The first conjecture was settled negatively by Junge and Pisier [10]. The second one is still open. In fact, Kirchberg showed that the second one is equivalent to the above Connes conjecture.

Theorem 2.7. *The following are equivalent:*

(i) *Connes's conjecture has a positive solution.*
(ii) *$(\mathcal{C}, \bar{\mathcal{C}})$ is nuclear.*
(iii) *For any $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in \mathcal{C}$ we have*

$$\| \sum_{j=1}^{n} x_j \otimes \bar{x}_j \|_{\min} = \| \sum_{j=1}^{n} x_j \otimes \bar{x}_j \|_{\max}$$

Equivalence between (ii) and (iii) is due to Haagerup. Kirchberg's conjecture is also equivalent to the *QWEP conjecture*: whether any C*-algebra is a quotient of a C*-algebra with the WEP. We refer to the reader to Ozawa's survey [17] for this topic.

In the rest of this section, we explain a connection between these conjectures with GT. Let $U_j \in \mathcal{C}$ be the unitary corresponding to the j-th generator of $\mathbb{F}_\infty$ and set $E := \overline{\mathrm{span}}\{I, U_j \mid j \in \mathbb{N}\} \subset \mathcal{C}$. Note that E is isometrically isomorphic to ℓ_1 via the mapping $U_j \mapsto e_j$.

Theorem 2.8 (Ozawa [18]). *The following are equivalent:*

- *$(\mathcal{C}, \mathcal{C})$ is nuclear*
- *For any $n \in \mathbb{N}$ and any $[a_{ij}] \in M_n(\mathbb{C})$, it follows that $\| \sum_{i,j=1}^n a_{ij} U_i \otimes \bar{U}_j \|_{\min} = \| \sum_{i,j=1}^n a_{ij} U_i \otimes \bar{U}_j \|_{\max}$, equivalently $\| \cdot \|_{\min} = \| \cdot \|_{\max}$ on $E \otimes_{\mathrm{alg}} \bar{E}$.*
- *Same holds for all $[a_{ij}] \geq 0$.*

On the other hand, GT tells us that these two norms on $E \otimes_{\mathrm{alg}} E$ are equivalent:

Proposition 2.9. *For any $a \in E \otimes_{\mathrm{alg}} \bar{E}$, it follows that $\|a\|_{\max} \leq K_G^{\mathbb{C}} \|a\|_{\min}$. Moreover, we have $\|a\|_{\max} \leq k_G^{\mathbb{C}} \|a\|_{\min}$ for $a \geq 0$. (Note that $k_G^{\mathbb{C}} = 4/\pi$).*

Proof. Let $a = \sum_{i,j=1}^n a_{ij} U_i \otimes \bar{U}_j$ be given. We also denote by a the matrix $[a_{ij}] \in \mathbb{M}_n(\mathbb{C})$. For any unitary representations $\pi, \sigma : \mathbb{F}_\infty \to \mathbb{B}(H)$ whose ranges commute, and any unit vectors $\xi, \eta \in S_H$, we have $|\langle \sum a_{ij}\pi(U_i)\sigma(U_j)\xi, \eta\rangle| = |\sum a_{ij}\langle \pi(U_i)\xi, \sigma(U_j)^*\eta\rangle| \le \|a\|_h$. Thus, it follows that $\|a\|_{\max} \le \|a\|_h$. On the other hand, for any $s_i, t_j \in S_{\mathbb{C}}$, if we set $\pi(U_i) := s_i$ and $\sigma(U_j) := t_j$, then we have $|\sum a_{ij}s_it_j| = \|\sum a_{ij}\pi(U_i) \otimes \sigma(U_j)\|_{\min} \le \|a\|_{\min}$, which implies $\|a\|_\vee \le \|a\|_{\min}$. Therefore, it follows from GT that $\|a\|_{\max} \le \|a\|_h \le K_G^{\mathbb{C}}\|a\|_\vee \le K_G^{\mathbb{C}}\|a\|_{\min}$. □

Proposition 2.10 (Tsirelson [28]). *For any $n \ge 1$ and $a = [a_{ij}] \in M_n(\mathbb{R})$, it follows that $\|a\|_{\max} = \|a\|_{\min}$.*

Proof. Since $\|a\|_{\min} \le \|a\|_{\max} \le \|a\|_h$ holds, it suffices to show that $\|a\|_h \le \|a\|_{\min}$. Since a_{ij} is real, we have $\|a\|_h = \sup\{|\sum a_{ij}\langle x_i, y_j\rangle| \mid x_i, y_j \in S_H\}$, where H is a *real* Hilbert space. We claim that for given $x_i, y_j \in S_H$, there exist a finite dimensional Hilbert space $\mathcal{F}$, selfadjoint unitaries u_i, v_j on $\mathcal{F}$, and a unit vector $\Omega \in \mathcal{F}$ satisfying that $u_iv_j = v_ju_i$ and $\langle u_iv_j\Omega, \Omega\rangle = \langle x_i, y_j\rangle$ for all i, j. This can be confirmed as follows. We can regard $x_i, y_j \in \mathbb{R}^n$ for all i, j. Set $K := \mathbb{C}^n$ and let $\mathcal{F} := \mathbb{C}\Omega \oplus \bigoplus_{k=1}^n K^{\wedge k}$ be the antisymmetric Fock space over K. For each $x \in K$, we define $c_x, d_x : \mathcal{F} \to \mathcal{F}$ by $c_x y := x \wedge y$ and $d_x y := y \wedge x$. Then, $u_i := c_{x_i} + c_{x_i}^*$ and $v_j := d_{y_j} + d_{y_j}^*$ are selfadjoint unitaries satisfying $u_iv_j = v_ju_i$[1]. We also have $\langle u_iv_j\Omega, \Omega\rangle = \langle x_i, y_j\rangle$. This proves the claim.

Since $\mathcal{F}$ is finite dimensional, the C*-algebra $B(\mathcal{F})$ is nuclear, and hence we have

$$|\sum a_{ij}\langle x_i, y_j\rangle| = |\sum a_{ij}\langle u_iv_j\Omega, \Omega\rangle| \le \|\sum a_{ij}u_i \otimes v_j\|_{\max} = \|\sum a_{ij}u_i \otimes v_j\|_{\min} \le \|a\|_{\min}.$$

Since x_i, y_j are arbitrary, we have $\|a\|_h \le \|a\|_{\min}$. □

3. Schur multipliers

For a given $\varphi = [\varphi_{ij}] \in M_n(\mathbb{K})$ the Schur multiplier $M_\varphi : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ is defined by $M_\varphi[a_{ij}] = [\varphi_{ij}a_{ij}]$. Let $\mathcal{S} := \{[s_it_j] \in M_n(\mathbb{K}) \mid s_i, t_j \in S_{\mathbb{K}}\}$. For each $\varphi = [s_it_j] \in \mathcal{S}$, we have $M_\varphi a = D_s a D_t$, where $D_s = \mathrm{diag}(s_1, \dots, s_n)$ and $D_t = \mathrm{diag}(t_1, \dots, t_n)$ are diagonal unitaries. Thus, we have $\|M_\varphi a\| = \|a\|$. For $a, b \in M_n$ we set $\langle a, b\rangle = \sum_{i,j} a_{ij}b_{ij}$. The next theorem is essentially due to Grothendieck.

Theorem 3.1. *The following hold true:*

(i) *For any $a \in M_n(\mathbb{K})$, we have $\|a\|_h = \sup_{\|M_\varphi\| \le 1} |\langle \varphi, a\rangle|$.*
(ii) *If $\|M_\varphi\| \le 1$ holds, then we have $\varphi \in K_G \overline{\mathrm{conv}}\, \mathcal{S}$.*

K_G is the best constant satisfying (ii). We need the next lemma, which will be proven in the next section (see Remark 4.7 below).

Lemma 3.2. *For any $a \in M_n$, $\|a\|_h \le 1$ if and only if $\|[\lambda_i^{-1/2}a_{ij}\mu_j^{-1/2}]\|_{M_n} \le 1$ for some $\lambda_j, \mu_j \ge 0$ with $\sum_{j=1}^n \lambda_j = \sum_{j=1}^n \mu_j = 1$ with the convention $0/0 := 0$.*

Proof of Theorem 3.1. We prove (i): For any $a \in M_n$ with $\|a\|_h = 1$, we can find $\lambda_i, \mu_j > 0$ such that $\sum \lambda_i = \sum \mu_j = 1$ and $\|[\lambda_i^{-1/2}a_{ij}\mu_j^{-1/2}]\|_{M_n} \le 1$. For any $\varphi \in M_n$, we have

$$\begin{aligned}
|\sum a_{ij}\varphi_{ij}| &= |\sum \lambda_i^{1/2}(\lambda_i^{-1/2}a_{ij}\mu_j^{-1/2}\varphi_{ij})\mu_j^{1/2}| \\
&= \|[\lambda_1^{1/2} \cdots \lambda_n^{1/2}][\lambda_i^{-1/2}a_{ij}\mu_j^{-1/2}\varphi_{ij}][\mu_1^{1/2} \cdots \mu_n^{1/2}]^T\| \\
&\le \|M_\varphi\|\|[\lambda_i^{-1/2}a_{ij}\mu_j^{-1/2}]\|_{M_n} \\
&\le \|M_\varphi\|,
\end{aligned}$$

[1]We use the fact that $\langle x_i, y_j\rangle$ is real here. For example, compare $u_iv_j\Omega = u_iy_j = x_i \wedge y_j + \langle y_j, x_i\rangle\Omega$ with $v_ju_i\Omega = v_jx_i = x_i \wedge y_j + \langle x_i, y_j\rangle\Omega$.

which implies $\sup_{\|M_\varphi\|\le 1} |\langle a, \varphi\rangle| \le \|a\|_h$. Conversely, for any $x_i, y_j \in S_H$, letting $\psi = [\langle x_i, y_j\rangle] \in M_n$ we have $|\sum_{i,j} a_{ij}\langle x_i, y_j\rangle| = |\langle a, \psi\rangle|$. Since it is not hard to see that $\|M_\psi\| \le 1$, we obtain $\|a\|_h \le \sup_{\|M_\varphi\|\le 1} |\langle a, \varphi\rangle|$.

We show (ii): By (i) and GT, we have

$$\sup_{\|M_\varphi\|\le 1} |\langle a, \varphi\rangle| \le K_G \sup_{\psi\in\mathcal{S}} |\langle a, \psi\rangle|.$$

Using the Hahn–Banach separation theorem, we get $\varphi \in K_G\,\overline{\mathrm{conv}}\,\mathcal{S}$. □

4. GT as a factorization theorem

A *state* on a C*-algebra A is a linear functional $f : A \to \mathbb{K}$ satisfying that $f(x^*x) \ge 0$ for $x \in A$ and $\|f\| = 1$. By the Gelfand–Naimark duality, every (unital) commutative C*-algebra is isomorphic to $C(S)$ for some compact Hausdorff space S. Note that there is a one-to-one correspondence between states on $C(S)$ and Radon probability measures on S.

Theorem 4.1 (GT/factorization)**.** *Let A and B be commutative* C**-algebras. Then, for any bounded bilinear form $\varphi : A\times B \to \mathbb{K}$, there exist states f and g on A and B respectively, such that*

$$|\varphi(x, y)| \le Cf(|x|^2)^{1/2} g(|y|^2)^{1/2} \quad \textit{for } a \in A, b \in B$$

with $C = K_G^{\mathbb{K}}\|\varphi\|$.

Remark 4.2. The above theorem says that every bounded linear map $u : C(S) \to C(T)^*$ factors through a Hilbert space, where S and T are compact Hausdorff spaces. Let φ be the bilinear form on $C(S) \times C(T)$ defined by $\varphi(x, y) = \langle u(x), y\rangle$ with the dual pairing $\langle\cdot,\cdot\rangle$ between $C(T)^*$ and $C(T)$. Theorem 4.1 says that there exist probability measures λ, μ on S, T, respectively satisfying that $|\langle u(x), y\rangle| \le C\|x\|_{L_2(\lambda)}\|y\|_{L_2(\mu)}$. Thus, if $J_\lambda : C(S) \to L_2(\lambda)$ and $J_\mu : C(T) \to L_2(\mu)$ denote the canonical (norm 1) maps, then there exists a linear map $v : L_2(\lambda) \to L_2(\mu)^*$ with $\|v\| \le C$ such that u is factorized as $u = J_\mu^* v J_\lambda$.

$$\begin{array}{ccc} C(S) & \xrightarrow{\ u\ } & C(T)^* \\ {\scriptstyle J_\lambda}\big\downarrow & & \big\uparrow{\scriptstyle J_\mu^*} \\ L_2(\lambda) & \xrightarrow{\ v\ } & L_2(\mu)^* \end{array}$$

Remark 4.3. GT says that any Banach space E with *isometric* embeddings $v : E \to L_1$ and $w : E^* \to L_1$ is *isomorphic* to a Hilbert space. Indeed, if we apply GT for $u = wv^* : L_\infty \to E^* \subset L_1 \subset L_\infty^*$, this map factors through a Hilbert space. Since v^* is surjective, E^* (and so E) is isomorphic to a Hilbert space. Note that the isomorphism obtained here is possibly not isometric. Indeed, Schneider [27] gave a counterexample, a finite dimensional *real* Banach space E such that $E \subset L_1$ and $E^* \subset L_1$ isometrically but E is not isometric to a Hilbert space. Curiously, the complex case apparently remains open.

Example 4.4. Consider the case when $A = C(S)$ and $B = C(T)$ with $S = T = \{1, \dots, n\}$. For any $a \in M_n$, $\|a\|_\vee$ equals to the norm of $\varphi : A \times B \to \mathbb{K}$ defined by $\varphi(x, y) = \sum_{s,t=1}^n a_{st}x(s)y(t)$. Then, the above theorem says that there exist probability vectors $\lambda = (\lambda_1, \cdots \lambda_n), \mu = (\mu_1, \dots, \mu_n)$ such that

$$|\sum_{s,t} a_{st}x_s y_t| \le C(\sum_s \lambda_s|x(s)|^2)^{1/2}(\sum_t \mu_t|y(t)|^2)^{1/2}.$$

We will show that Theorem 4.1 is equivalent to the following theorem:

Theorem 4.5. *Let $\varphi : A \times B \to \mathbb{K}$ be as in Theorem 4.1. Then, for any $m \geq 1$, $x_1, \dots, x_m \in A$, and $y_1, \dots, y_m \in B$, we have*

$$|\sum_{j=1}^m \varphi(x_j, y_j)| \leq C \| \sum_{j=1}^m |x_j|^2 \|^{1/2} \| \sum_{j=1}^m |y_j|^2 \|^{1/2}.$$

We first consider the case when $S = T = \{1, \dots, n\}$. Let $\varphi : C(S) \times C(T) \to \mathbb{K}$ be a given bilinear form and $a \in \mathbb{M}_n$ be the matrix given by $\sum_{s,t=1}^n a_{st} x(s) y(t) = \varphi(x, y)$ for $x \in C(S)$ and $y \in C(T)$. Note that $\|\varphi\| = \|a\|_\vee$. Fix $m \geq 1$ and $x_1, \dots, x_m \in A, y_1, \dots, y_m \in B$ arbitrarily. Set $X(s) := \sum_j x_j(s) e_j \in \ell_2$ and $Y(t) := \sum_j y_j(t) e_j \in \ell_2$. Since $\|a\|_h = \sup\{ | \sum a_{ij} \langle z_i, w_j \rangle | \mid z_i, w_j \in B_H \}$ holds, we have $| \sum_{s,t=1}^n a_{st} \langle X(s), Y(t) \rangle | \leq \|a\|_h \sup_s \|X(s)\|_{\ell_2} \sup_t \|Y(t)\|_{\ell_2}$. Thus, Theorem 1.1 tells us that

$$|\sum_{s,t=1}^n a_{st} \langle X(s), Y(t) \rangle | \leq K_G^{\mathbb{K}} \|a\|_\vee \sup_s \|X(s)\|_{\ell_2} \sup_t \|Y(t)\|_{\ell_2}.$$

This is equivalent to

$$|\sum_{j=1}^m \varphi(x_j, y_j)| \leq K_G^{\mathbb{K}} \|\varphi\| \sup_s \left(\sum_{j=1}^m |x_j(s)|^2 \right)^{1/2} \sup_t \left(\sum_{j=1}^m |y_j(t)|^2 \right)^{1/2}.$$

Therefore, in this case Theorem 4.5 follows from Theorem 1.1. In fact, we can reduce Theorem 4.5 to the finite dimensional case (c.f. [19]). Thus, we only have to show that Theorem 4.5 $\Rightarrow$ Theorem 4.1. (The converse implication easily follows from the Cauchy–Schwarz inequality.) We need the following variant of the Hahn–Banach theorem:

Theorem 4.6. *Let S be a set and $\mathcal{F} \subset \ell_\infty(S, \mathbb{R})$ be a convex cone satisfying $\sup_{s \in S} f(s) \geq 0$ for all $f \in \mathcal{F}$. Then, there exists a net λ_α of finitely supported probability measures on S such that $\lim_\alpha \int f d\lambda_\alpha \geq 0$ for all $f \in \mathcal{F}$.*

Proof. By assumption, we have $\mathcal{F} \cap \{ g \in \ell_\infty(S, \mathbb{R}) \mid \sup_{s \in S} g(s) < 0 \} = \emptyset$. Thus, by the Hahn–Banach separation theorem, we can find $\lambda \in \ell^\infty(S, \mathbb{R})^*$ such that $\lambda(f) \geq 0$ for $f \in \mathcal{F}$. Since λ can be approximated by finitely supported probability measures on S in the weak*-topology, we are done. □

Proof of Theorem 4.5 $\Rightarrow$ Theorem 4.1. Let $A = C(S)$ and $B = C(T)$. Let $m \in \mathbb{N}$, $x = (x_i) \in A^m$, $y = (y_j) \in B^m$ be arbitrary. By assumption and the arithmetic-geometric mean inequality, we have

$$|\sum_{j=1}^m \varphi(x_j, y_j)| \leq \frac{1}{2} C \sup_{S \times T} (\sum_{i=1}^m |x_i(s)|^2 + \sum_{j=1}^m |y_j(t)|^2).$$

Define $f_{x,y} \in \ell_\infty(S \times T, \mathbb{R})$ by

$$f_{x,y}(s,t) := \frac{C}{2} (\sum_i |x_i(s)|^2 + \sum_j |y_j(t)|^2) - | \sum_i \varphi(x_i, y_i)|$$

Then, $\mathcal{F} := \{ f_{x,y} \mid m \in \mathbb{N}, (x,y) \in A^m \times B^m \}$ forms a convex cone satisfying that $\sup_{S \times T} f_{x,y}(s,t) \geq 0$ for all $f_{x,y} \in \mathcal{F}$. By Theorem 4.6 we can find a net λ_α of finitely supported probability measures on $S \times T$ satisfying

$$|\varphi(x,y)| \leq \frac{C}{2} \lim_\alpha \left(\int_{S \times T} |x(s)|^2 d\lambda_\alpha + \int_{S \times T} |y(t)|^2 d\lambda_\alpha \right).$$

Then, there exist probability measures $\mathbb{P}$ and $\mathbb{Q}$ on S and T, respectively such that $\lim_\alpha \int_{S\times T} |x(s)|^2 d\lambda_\alpha = \int_S |x|^2 d\mathbb{P}$ and $\lim_\alpha \int_{S\times T} |y(t)|^2 d\lambda_\alpha = \int_T |y|^2 d\mathbb{Q}$, and hence we get

$$|\varphi(x,y)| = |\varphi(tx, \frac{y}{t})| \le \frac{C}{2}\left(t^2\|x\|_{L_2(\mathbb{P})} + \frac{1}{t^2}\|y\|_{L_2(\mathbb{Q})}\right)$$

for all $t>0$. Since $\sqrt{ab} = \inf_{t>0} \frac{1}{2}(ta + \frac{b}{t})$ holds for any $a, b \in [0,\infty)$, we obtain $|\varphi(x,y)| \le C\|x\|_{L_2(\mathbb{P})}\|y\|_{L_2(\mathbb{Q})}$.

□

Remark 4.7. We can now prove Lemma 3.2. Let $a = [a_{ij}]$ be a given matrix and $\varphi : \ell_\infty^n \times \ell_\infty^n \to \mathbb{K}$ be as in Example 4.4. Then, $\|a\|_h \le 1$ holds if and only if the assertion of Theorem 4.5 holds with $C = 1$. Let λ, μ be probability measures on $\{1,\dots,n\}$ as in Theorem 4.1. Then, we can check that $\|[\lambda_i^{-1/2} a_{ij} \mu_j^{-1/2}]_{ij}\|_{M_n} \le 1$ with the convention $0/0 = 0$.

Theorem 4.8 (Little GT/factorization)**.** *Let A be a commutative* C*-*algebra. For any bounded linear map u from A to a Hilbert space H, there exists a state f on A such that*

$$\|u(x)\| \le \sqrt{k_G}\|u\| f(|x|^2)^{1/2} \quad \textit{for } x \in A.$$

Proof. Set $\varphi(x,y) := \langle u(x), u(y)\rangle$ for $x, y \in A$. Since $\|\varphi\| = \|u\|^2$ holds, Theorem 4.1 says that there exists a probability measure $\mathbb{P}$ on S, where $A = C(S)$, such that $\|u(x)\|^2 \le C\|x\|^2_{L_2(\mathbb{P})}$. When $A \cong \ell_\infty^n$, the matrix $[\langle u(e_i), u(e_j)\rangle]_{ij}$ is positive definite. Thus, we get $\|u(x)\|^2 \le k_G\|u\|^2\|x\|^2_{L_2(\mathbb{P})}$ □

Note that $k_G \le K_G$. We next see that

$$k_G \ge \frac{1}{\|g\|_1^2} = \begin{cases} \pi/2 & \mathbb{K} = \mathbb{R}, \\ 4/\pi & \mathbb{K} = \mathbb{C}, \end{cases}$$

where g is a standard Gaussian random variable. Let $g_1, g_2, \dots$ be an i.i.d. sequence of copies of g on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $G := \overline{\mathrm{span}}\{g_i \mid i \ge 1\}$ and $P : L_2 \to G$ be the orthogonal projection. Consider a natural map $J : L_2 \to L_1$. For each $x \in G$, we have $\|Jx\|_1 = \|g\|_1\|x\|_2$. Define the bilinear form $\varphi : L_\infty \times L_\infty \to \mathbb{K}$ by $\varphi(x,y) := \langle x, Py\rangle$. Then, we have $\|\varphi\| = \|g\|_1^2$. By the fact that $\|\varphi\|_h := \inf\{C \text{ in Theorem 4.5}\} \ge 1$, we have $1 \le \|\varphi\|_h \le k_G\|\varphi\| = k_G\|g\|_1^2$, and hence $k_G \ge \|g\|_1^{-2}$.

5. Non-commutative GT

Grothendieck conjectured a non-commutative analogue of Theorem 4.1. This was proved by Pisier [20] under an approximation assumption. The following optimal form was proved by Haagerup [5].

Theorem 5.1 (Non-commutative GT)**.** *Let A and B be* C*-*algebras and $\varphi : A \times B \to \mathbb{C}$ be a bounded bilinear form. Then, there exist states f_1, f_2 on A and g_1, g_2 on B such that*

$$|\varphi(x,y)| \le C(f_1(x^*x) + f_2(xx^*))^{1/2}(g_1(y^*y) + g_2(yy^*))^{1/2} \quad \textit{for } x \in A, y \in B$$

with $C = \|\varphi\|$.

Note that this theorem implies the original GT with constant ≤ 2. In [7] it was proved that the constant 1 is optimal for this non-commutative GT. As in the commutative case, we can show that Theorem 5.1 is equivalent to the following theorem:

Theorem 5.2. *Let $\varphi : A \times B \to \mathbb{C}$ be as above. Then, for any $m \in \mathbb{N}$, $x = (x_j)_j \in A^m$, and $y = (y_j) \in B^m$ one has*

$$|\sum_j \varphi(x_j, y_j)| \le C\{\|\sum_j x_j^* x_j\| + \|\sum_j x_j x_j^*\|\}^{1/2}\{\|\sum_j y_j^* y_j\| + \|\sum_j y_j y_j^*\|\}^{1/2}.$$

Let us introduce the following notations. For $m \geq 1$, $x = (x_j)_{j=1}^m \in A^m$, we define the *column* and *row* norms by

$$\|x\|_C := \|\sum_j x_j^* x_j\|^{1/2}, \quad \|x\|_R := \|\sum_j x_j x_j^*\|^{1/2}.$$

For $p \geq 1$, we denote by $C_p(\varphi)$ the best constant $C > 0$ (possibly $C = +\infty$) such that

$$|\sum_j \lambda_j \varphi(x_j, y_j)| \leq C \left\{ \|\sum_j \lambda_j |x_j|^p\| + \|\sum_j \lambda_j |x_j^*|^p\| \right\}^{1/p} \left\{ \|\sum_j \lambda_j |y_j|^p\| + \|\sum_j \lambda_j |y_j^*|^p\| \right\}^{1/p}$$

holds for all $m \in \mathbb{N}$ and $x \in A^m, y \in B^m$ and $\lambda_j \geq 0$ with $\sum_{j=1}^m \lambda_j = 1$. Also we set $C_\infty(\varphi) := \|\varphi\|$. Our goal is to show that $C_2(\varphi) \leq C_\infty(\varphi)$. This is proved in the following steps.

Step 1: Show that $C_2(\varphi) \leq C_4(\varphi)$.

Step 2: Show that $C_4(\varphi) \leq \sqrt{C_2(\varphi) C_\infty(\varphi)}$.

Step 3: Show that $C_2(\varphi) < \infty$.

Indeed, Step 1 and Step 2 leads to $C_2(\varphi)^2 \leq C_2(\varphi) C_\infty(\varphi)$, and then Step 3 enables us to divide this inequality by $C_2(\varphi)$; giving $C_2(\varphi) \leq C_\infty(\varphi)$.

Proof of Step 1. Fix $m \geq 1$, $x = (x_j) \in A^m$ and $y = (y_j) \in B^m$ arbitrarily. Define vector-valued continuous functions $S_x : \mathbb{T}^m \to A$ and $T_y : \mathbb{T}^m \to B$ by $S_x(z) := \sum_j z_j x_j$ and $T_y(z) := \sum_j \bar{z}_j y_j$ for $z = (z_j) \in \mathbb{T}^m$. Let μ be the Haar probability measure on $\mathbb{T}^m$. By the definition of $C_4(\varphi)$, it follows that

$$|\mathbb{E}[\varphi(S_x, T_y)]| \leq C_4(\varphi) \left\{ \|\mathbb{E}[|S_x|^4]\| + \|\mathbb{E}[|S_x^*|^4]\| \right\}^{1/4} \left\{ \|\mathbb{E}[|T_y|^4]\| + \|\mathbb{E}[|T_y^*|^4]\| \right\}^{1/4}.$$

We observe that $|S_x|^4 = (S_x^* S_x)^2 = (\sum_{j=1}^m x_j^* x_j + \sum_{k \neq l} \bar{z}_k z_l x_k^* x_l)^2$. By the orthogonality of $(z_j)_j$ (with respect to μ), we have

$$\mathbb{E}[|S_x|^4] = \int_{\mathbb{T}^m} (\sum_j x_j^* x_j + \sum_{k \neq l} \bar{z}_k z_l x_k^* x_l)^2 d\mu = (\sum_j x_j^* x_j)^2 + \sum_{k \neq l} x_k^* x_l x_l^* x_k.$$

Thus, the inequality that $\|\sum_{k \neq l} x_k^* x_l x_l^* x_k\| \leq \|x\|_R^2 \|x\|_C^2$ implies that

$$\|\mathbb{E}[|S_x|^4]\| + \|\mathbb{E}[|S_x^*|^4]\| \leq (\|x\|_C^2 + \|x\|_R^2)^2.$$

Since the same inequality holds for y and T, we have

$$|\mathbb{E}[\varphi(S_x, T_y)]| \leq C_4(\varphi) (\|x\|_C^2 + \|x\|_R^2)^{1/2} (\|y\|_C^2 + \|y\|_R^2)^{1/2}.$$

On the other hand, for any $\lambda = (\lambda_j)$ with $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$, one has

$$|\sum_j \lambda_j \varphi(x_j, y_j)| = |\int \sum_{i,j} \lambda_i^{1/2} \lambda_j^{1/2} z_i \bar{z}_j \varphi(x_i, y_j) d\mu| = |\mathbb{E}[\varphi(S_{\lambda^{1/2} \cdot x}, T_{\lambda^{1/2} \cdot y})]|,$$

where $\lambda^{1/2} \cdot x = (\lambda_j^{1/2} x_j)$. Thus, the above estimate implies that

$$|\sum_j \lambda_j \varphi(x_j, y_j)| \leq C_4(\varphi) (\|\lambda^{1/2} \cdot x\|_C^2 + \|\lambda^{1/2} \cdot x\|_R^2)^{1/2} (\|\lambda^{1/2} \cdot y\|_C^2 + \|\lambda^{1/2} \cdot y\|_R^2)^{1/2},$$

so we have $C_2(\varphi) \leq C_4(\varphi)$. □

Proof of Step 2. Take $x = (x_j) \in A^m$ and $y = (y_j) \in B^m$ and $\lambda = (\lambda_j)$ arbitrarily. For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, we set $f_j(z) = u_j |x_j|^z \in A$ and $g_j(z) = v_j |y_j|^z \in B$, where $x_j = u_j |x_j|$ and $y_j = v_j |y_j|$ are polar decompositions. Then, f_j's and g_j's are analytic in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, and hence so is $h(z) := \sum_j \lambda_j \varphi(f_j(z), g_j(z))$. We observe that $f_j(z)^* f_j(z) = (x_j^* x_j)^{\operatorname{Re} z}$,

$f_j(z)f_j(z)^* = (x_jx_j^*)^{\operatorname{Re} z}$, $g_j(z)^*g_j(z) = (y_j^*y_j)^{\operatorname{Re} z}$ and $g_j(z)g_j(z)^* = (y_jy_j^*)^{\operatorname{Re} z}$. Thus, for any $t \in \mathbb{R}$ we have

$$|h(2+it)| \le C_2(\varphi)\{\|\sum_j \lambda_j (x_j^*x_j)^2\| + \|\sum_j \lambda_j (x_jx_j^*)^2\|\}^{1/2} \times \{\|\sum_j \lambda_j (y_j^*y_j)^2\| + \|\sum_j \lambda_j (y_jy_j^*)^2\|\}^{1/2}.$$

For $\varepsilon \in (0,1)$, we also have

$$|h(\varepsilon+it)| \le C_\infty(\varphi)\sum_j \lambda_j \|x_j\|^\varepsilon \|y_j\|^\varepsilon.$$

Applying the three line theorem (see, e.g. [1, Lemma 1.2.2]) to h and the strip $\{z \in \mathbb{C} \mid \varepsilon < \operatorname{Re} z < 2\}$, we obtain

$$|h(1)| \le \sup_{t\in\mathbb{R}} |h(\varepsilon+it)|^{1/(2-\varepsilon)} \sup_{s\in\mathbb{R}} |h(2+is)|^{(1-\varepsilon)/(2-\varepsilon)}.$$

Taking the limit $\varepsilon \to 0$ we have

$$|\sum_j \lambda_j \varphi(x_j,y_j)| \le \sqrt{C_\infty(\varphi)C_2(\varphi)}\{\|\sum_j \lambda_j |x_j|^4\| + \|\sum_j \lambda_j |x_j^*|^4\|\}^{1/4} \times \{\|\sum_j \lambda_j |y_j|^4\| + \|\sum_j \lambda_j |y_j^*|^4\|\}^{1/4},$$

which implies $C_4(\varphi) \le \sqrt{C_\infty(\varphi)C_2(\varphi)}$. □

Finally, we briefly explain the outline of Step 3 (see [5, Section 3] for details). It suffices to show that $C_2(\varphi) < \infty$ when A and B are von Neumann algebras and $\|\varphi\| = \varphi(1,1) = 1$. (By considering second duals and ultraproducts, we may assume that $\|\varphi\| = \varphi(u,v)$ for some unitaries u, v. Then, we can replace φ by $\varphi'(a,b) := \varphi(au,bv)$.) Then, $f(x) := \varphi(x,1)$ and $g(y) := \varphi(1,y)$ for $x \in A$ and $y \in B$ define states on A and B, respectively. For any $a = a^* \in A$ and $b = b^* \in B$, it is not so hard to show that $|\operatorname{Re}\varphi(a,b)| \le f(a^2)^{1/2}g(b^2)^{1/2}$. For the imaginary part of $\varphi(a,b)$, the similar inequality does not hold in general. However, using spectral resolutions of a and b we can show that $|\operatorname{Im}\varphi(a,b)| \le 2f(a^4)^{1/4}g(b^4)^{1/4}$. Thus, by a similar argument as in Step 1 (using Rademacher functions instead of z_j's), we can find states f' and g' in such a way that

$$|\operatorname{Im}\varphi(a,b)| \le 4f'(a^2)^{1/2}g'(b^2)^{1/2}.$$

Therefore, letting $f'' := f/5 + 4f'/5$ and $g'' := g/5 + 4g'/5$ we have

$$|\varphi(x,y)| \le \frac{5}{2}\varphi''(x^*x + xx^*)^{1/2}\psi''(y^*y + yy^*)^{1/2}$$

for all $x \in A$, $y \in B$. This implies $C_2(\varphi) < \infty$.

Remark 5.3. The ideas for steps 1 and 2 are already in [20], but step 3 is the main new contribution from [5].

Theorem 5.4 (Non-commutative little GT)**.** *For any bounded linear map u from a C^*-algebra A into a Hilbert space H, there exist states f_1 and f_2 on A such that*

$$\|u(x)\| \le \|u\|(f_1(x^*x) + f_2(xx^*))^{1/2} \quad \textit{for } x \in A.$$

6. GT for operator spaces

An *operator space* E is a closed subspace of $B(H)$ with a Hilbert space H. Then, for any $n \geq 1$, we can induce the norm on $M_n(E)$ from $M_n(B(H)) \cong B(H^{\oplus n})$. Let F be another operator space and $u : E \to F$ be a linear map. For each $n \in \mathbb{N}$, we define $u_n : M_n(E) \to M_n(F)$ by $u_n([x_{ij}]) = [u(x_{ij})]$. The linear map u is said to be *completely bounded* (c.b. for short) if $\|u\|_{cb} := \sup_{n\geq 1} \|u_n : M_n(E) \to M_n(F)\| < \infty$ holds.

Definition 6.1. Let E, F and G be operator spaces and $\varphi : E \times F \to G$ be a bilinear map. For each $n \in \mathbb{N}$, we define $\varphi_n : M_n(E) \times M_n(F) \to M_{n^2}(G)$ by $\varphi_n([a_{ij}], [b_{pq}]) := [\varphi(a_{ij}, b_{pq})]_{(i,j),(p,q)}$. We say that φ is *jointly completely bounded* (j.c.b. for short) if $\|\varphi\|_{jcb} := \sup_n \|\varphi_n\| < \infty$.

For any operator space E, its dual space E^* has a natural operator space structure. If F is an operator space and $\varphi : E \times F \to \mathbb{C}$ is a bounded bilinear form, then one has $\|\varphi\|_{jcb} = \|u_\varphi\|_{cb} = \|v_\varphi\|_{cb}$, where $u_\varphi : E \to F^*$ and $v_\varphi : F \to E^*$ are linear maps defined by $\varphi(x, y) = \langle u_\varphi(x), y\rangle_{F^*\times F} = \langle x, v_\varphi(y)\rangle_{E\times E^*}$ for $x \in E, y \in F$.

Theorem 6.2 (Operator space GT). *Let A and B be* C*-*algebras and $\varphi : A \times B \to \mathbb{C}$ be a j.c.b. bilinear form. Then, there exist states f_1, f_2 and g_1, g_2 on A and B, respectively such that*

$$|\varphi(x, y)| \leq C \left(f_1(x^*x)^{1/2} g_1(yy^*)^{1/2} + f_2(xx^*)^{1/2} g_2(y^*y)^{1/2} \right)$$

for all $x \in A, y \in B$ with $C = \|\varphi\|_{jcb}$.

Pisier and Shlyakhtenko [25] showed this inequality for exact (see Definition 7.1 below) C*-algebras using type III von Neumann algebras. Haagerup and Musat [8] generalized it for arbitrary C*-algebras using different type III von Neumann algebras. Very recently, Regev and Vidick [26] gave a very short proof using ideas originating in quantum information theory.

The reader should compare this operator space GT with the non-commutative GT (Theorem5.1) for bounded bilinear forms. We note that if the above inequality holds for a given bilinear form φ, then φ is j.c.b. with $\|\varphi\|_{jcb} \leq 2C$. This follows from the next proposition.

Proposition 6.3. *Consider operator spaces $E \subset A$ and $F \subset B$ and a j.c.b. map $\varphi : E \times F \to \mathbb{C}$. For any states f, f' on A and g, g' on B, the following hold true:*

- (i) *If $|\varphi(x, y)| \leq f(xx^*)^{1/2} g(y^*y)^{1/2}$ holds for $(x, y) \in E \times F$, then we have $\|\varphi\|_{jcb} \leq 1$.*
- (ii) *If $|\varphi(x, y)| \leq f'(x^*x)^{1/2} g'(yy^*)^{1/2}$ holds for $(x, y) \in E \times F$, then we have $\|\varphi\|_{jcb} \leq 1$.*
- (iii) *If $|\varphi(x, y)| \leq f(xx^*)^{1/2} g(y^*y)^{1/2} + f'(x^*x)^{1/2} g'(yy^*)^{1/2}$ for $(x, y) \in E \times F$, then we have $\|\varphi\|_{jcb} \leq 2$.*

The assertion (iii) follows from the following lemma (credited to Pisier in [29]).

Lemma 6.4 ([29, Proposition 5.1]). *If the assumption of* (iii) *holds, then there exist φ_1, φ_2 such that $\varphi = \varphi_1 + \varphi_2$ and φ_1 and φ_2 satisfy the assumption of* (i) *and* (ii)*, respectively.*

Using Theorem 4.6 we can show the following proposition.

Proposition 6.5 ([22, Proposition 18.2]). *Let $E \subset A$ and $F \subset B$ be operator spaces and let φ be a bilinear form on $E \times F$. The following are equivalent:*

- (i) *For any $m \in \mathbb{N}$, $x = (x_j) \in E^m$, $y = (y_j) \in F^m$ and $t = (t_j)$ with $t_j > 0$, it follows that*

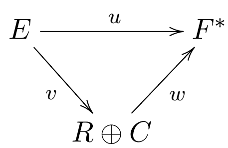
$$|\sum_j \varphi(x_j, y_j)| \leq \left(\|x\|_C \|y\|_R + \|t \cdot x\|_R \|t^{-1} \cdot y\|_C\right).$$

(ii) *There exist states f_1, f_2 on A and g_1, g_2 on B such that*

$$|\varphi(x,y)| \le \left(f_1(x^*x)^{1/2} g_1(yy^*)^{1/2} + f_2(xx^*)^{1/2} g_2(y^*y)^{1/2}\right)$$

for all $x \in E, y \in F$.

Remark 6.6. As in the classical case, the above inequalities are equivalent to a factorization property: Consider two operator spaces $R := \overline{\text{span}}\{e_{1j} \mid j \ge 1\}, C := \overline{\text{span}}\{e_{i1} \mid i \ge 1\} \subset B(\ell_2)$. Let E and F be separable operator spaces and $u : E \to F^*$ be a c.b. map. Suppose that the bilinear form φ on $E \times F$ associated to u satisfies the conditions in the previous proposition. Then, one has $\inf\{\|v\|_{cb}\|w\|_{cb}\} \le 2$, where the infimum runs over all factorizations of u through the operator space $R \oplus C \subset B(\ell^2 \oplus \ell^2)$:



Conversely, $\inf\{\|v\|_{cb}\|w\|_{cb}\} \le 1$ implies that φ satisfies the conditions in Proposition 6.5.

Lemma 6.7. *For any $n \in \mathbb{N}$ and any $t > 0$, there exists a matrix $L(t) \in M_n$ with positive matrix entries $L(t)_{pq} > 0$ satisfying the following:*

(i) $\sup_p \sum_{q=1}^n L(t)_{pq} \le 1$;

(ii) $\sup_q \sum_{p=1}^n L(t)_{pq} \le t^2$;

(iii) $|t^{-1}\langle L(t)\Phi_n, \Phi_n\rangle - 1| \le C(\log n)^{-1}\log(1+\max(t,t^{-1}))$,

where $\Phi_n = \frac{1}{z_n}\sum_{p=1}^n \frac{1}{\sqrt{p}} e_p \in \mathbb{C}^n$ with $z_n = (\sum_{p=1}^n p^{-1})^{1/2}$ and C is a universal constant.

Proof. Set $L(t)_{pq} := |[p-1,p) \cap [(q-1)t^2, qt^2)|$ for $p, q = 1, \dots, n$ and $t > 0$. Then, one can easily check the assertions (i) and (ii). The last assertion can be shown by calculus. □

The vector $\widehat{\Phi}_n := \frac{1}{z_n}\sum_{p=1}^n \frac{1}{\sqrt{p}} e_p \otimes e_p$ is known as "embezzlement state" in quantum information theory.

Proof of Theorem 6.2 by Regev–Vidick. Fix $n \ge 1$. Let $\{e_{pq}\}_{p,q=1}^n$ be a system of matrix units of M_n. Set $X_{pqj} := L(t_j)_{pq}^{1/2} e_{pq} \otimes x_j \in M_n \otimes A$ and $Y_{pqj} := t_j^{-1} L(t_j)_{pq}^{1/2} e_{pq} \otimes y_j \in M_n \otimes B$ for $1 \le j \le m$ and $1 \le p, q \le n$. We may assume that $\|\varphi\|_{jcb} = 1$. Let $\widehat{\Phi}_n \in \mathbb{C}^n \otimes \mathbb{C}^n$ be as above and ψ be the vector state on $M_n \otimes M_n$ associated with $\widehat{\Phi}_n$. Since $\psi \circ \varphi_n$ defines a contractive bilinear form on $M_n(A) \times M_n(B)$, we can apply the non-commutative GT for $X := (X_{pqj})_{p,q,j}$ and $Y := (Y_{pqj})_{p,q,j}$. Then, we obtain

$$|\sum_{j=1}^m \sum_{p,q=1}^n \psi \circ \varphi_n(X_{pqj}, Y_{pqj})| \le (\|X\|_R^2 + \|X\|_C^2)^{1/2}(\|Y\|_R^2 + \|Y\|_C^2)^{1/2}.$$

It follows from (i) and (ii) of Lemma 6.7 that $\|X\|_R \le \|x\|_R$, $\|X\|_C \le \|t\cdot x\|_C$, $\|Y\|_C \le \|y\|_C$, and $\|Y\|_R \le \|t^{-1}\cdot y\|_R$. For each $1 \le j \le m$, we have $\sum_{p,q} \psi \circ \varphi_n(X_{pqj}, Y_{pqj}) = \sum_{p,q}\langle \varphi_n(X_{pqj}, Y_{pqj})\widehat{\Phi}_n, \widehat{\Phi}_n\rangle = \varphi(x_j, y_j) t_j^{-1}\langle L(t_j)\Phi_n, \Phi_n\rangle$. Thus, by (iii) of Lemma 6.7, letting $n \to \infty$ we obtain $|\sum_{j=1}^m \varphi(x_j, y_j)| \le (\|x\|_R^2 + \|t\cdot x\|_C^2)^{1/2}(\|t^{-1}\cdot y\|_R^2 + \|y\|_C^2)^{1/2}$. Replacing x_j, y_j by $sx_j, s^{-1}y_j$ and t_j by $s't_j$ for some $s, s' > 0$, we have $|\sum_{j=1}^m \varphi(x_j, y_j)| \le \|x\|_R\|y\|_C + \|t\cdot x\|_C\|t^{-1}\cdot y\|_R$. □

Corollary 6.8. *Let E be an operator space. If there exist C^*-algebras A and B such that E and E^* can be embedded into A^* and B^* completely isometrically, then E is a subquotient of $R \oplus C$.*

Proof. By assumption and the operator space GT, E must be isomorphic to a Hilbert space and fortiori reflexive. Thus the embeddings $E \subset A^*$ and $E^* \subset B^*$ give a mapping $u : B^{**} \to E \subset A^*$. Applying the operator space GT again, u factors through $R \oplus C$. □

7. GT for exact operator spaces

It is natural to consider GT for bilinear forms on arbitrary Banach spaces, but the class of pairs of Banach spaces for which GT holds is known to be rather restrictive. However, for j.c.b. maps on exact operator spaces, we can show an analogue of GT.

Let E and F be operator spaces. We define the *c.b. distance* by

$$d_{cb}(E,F) := \inf\{\|u\|_{cb}\|u^{-1}\|_{cb} \mid u : E \to F \text{ isomorphism}\}.$$

When E and F are not isomorphic, we set $d_{cb}(E,F) = \infty$.

Definition 7.1. An operator space $X \subset B(H)$ will be called C*-exact* if for any finite dimensional subspace $E \subset X$, there exist $N \in \mathbb{N}$ and $F \subset M_N$ such that $d_{cb}(E,F) \leq C$. The *exact constant* of X is defined by $ex(X) := \inf\{C \mid X \text{ is } C\text{-exact}\}$.

Kirchberg showed that for any C*-algebra A its exact constant is 1 or ∞. More generally, it is known that the following are equivalent:

- A is exact.
- $ex(A) = 1$.
- $ex(A) < \infty$.
- Any embedding $\iota_A : A \hookrightarrow B(H)$ is nuclear, i.e., for any C*-algebra B, $\iota_A \otimes \mathrm{id}_B : A \otimes_{\max} B \to B(H) \otimes_{\max} B$ factors through $A \otimes_{\min} B$.

When A is separable, the above conditions are also equivalent to

- A is a subalgebra of a nuclear C*-algebra.
- A is a subalgebra of the Cuntz algebra $\mathcal{O}_2$.

For any operator space E with $\dim E = n$, we have $ex(F) \leq n$. Let $E_n = \mathrm{span}\{U_1, \ldots, U_n\} \subset \mathcal{C} = \mathrm{C}^*(\mathbb{F}_\infty)$, then one has $ex(E_n) \geq \frac{n}{2\sqrt{n-1}} \geq \frac{\sqrt{n}}{2}$. Thus, $\mathcal{C}$ is not exact (see Example 8.6 below).

The following is due to Junge and Pisier [10].

Theorem 7.2 (GT for exact operator spaces)**.** *If E and F are exact operator spaces and $\varphi : E \times F \to \mathbb{C}$ is a j.c.b. map, then for any $m \in \mathbb{N}$ and any $x = (x_j) \in E^m$, $y = (y_j) \in F^m$ one has*

$$|\sum_j \varphi(x_j, y_j)| \leq C(\|x\|_C^2 + \|x\|_R^2)^{1/2}(\|y\|_C^2 + \|y\|_R^2)^{1/2}$$

with $C = 2ex(E)ex(F)\|\varphi\|_{jcb}$.

This can be shown by using randam matrix techniques. Let $Y^{(N)}$ be an $N \times N$-Gaussian random matrix, i.e., its matrix entries form an i.i.d. sequence of $N(0, N^{-1})$ Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will denote by $Y^{(N)}(i,j)$ the (i,j)-entry of $Y^{(N)}$. Let $Y_1^{(N)}, Y_2^{(N)}, \ldots$ be an i.i.d. sequence of copies of $Y^{(N)}$. By the strong law of large numbers, $\tau_N(Y_i^{(N)}Y_j^{(N)})$ converges to $\delta_{i,j}$ almost surely. Thus, one has almost surely

$$|\sum_{j=1}^m \varphi(x_j, y_j)| = \lim_{N\to\infty} |\sum_{i,j=1}^m \varphi(x_i, y_j)\tau_N(Y_i^{(N)}Y_j^{(N)})|$$
$$\leq \limsup_{N\to\infty} \|\varphi\|_{jcb}\|\sum_{j=1}^m Y_j^{(N)} \otimes x_j\|\|\sum_{j=1}^m Y_j^{(N)} \otimes y_j\|.$$

Let $\mathcal{F}$ be the full Fock space over $H := \ell_2(\mathbb{N}\times\{1,2\})$, i.e., $\mathcal{F} = \mathbb{C}\oplus H\oplus H^{\otimes 2}\oplus\cdots$. For a given $h \in H$, we denote by $\ell(h)$ the left creation operator on $\mathcal{F}$. Let $\{e_{(i,j)}\}_{(i,j)\in\mathbb{N}\times\{1,2\}}$ be the canonical basis. Define "free circular" elements $c_j := \ell(e_{(j,1)}) + \ell(e_{(j,2)})^*$ for $j \in \mathbb{N}$. For any $m, K \in \mathbb{N}$ and $x = (x_j) \in (M_K)^m$, one can check that $\|\sum_{j=1}^m c_j \otimes x_j\| \le \|x\|_R + \|x\|_C$. Thus, Theorem 7.2 follows from Haagerup and Thorbjørnsen's result [9], which guarantees that $\limsup_{N\to\infty}\|\sum_{j=1}^m Y_j^{(N)}\otimes x_j\| \le \|\sum_{j=1}^m c_j\otimes x_j\|$ almost surely.

8. GT for subexponential operator spaces

In this section, we consider GT for more general operator spaces. We refer the reader to [24] for a reference of this section.

8.1. **Tight and Subexponential operator spaces.**

Definition 8.1. Let E be an operator space.

(i) E is said to be *C-tight* if for any $m \in \mathbb{N}$ and $x_1,\ldots,x_m \in E$, one has almost surely

$$\limsup_{N\to\infty}\mathbb{E}\|\sum_{j=1}^m Y_j^{(N)}\otimes x_j\| = \limsup_{N\to\infty}\|\sum_{j=1}^m Y_j^{(N)}\otimes x_j\| \le C(\|x\|_R^2+\|x\|_C^2)^{1/2}.$$

(ii) E is said to be *completely C-tight* if $M_n(E)$ is C-tight for all $n\in\mathbb{N}$.

Note that the first equality in (i) always holds by well known results on concentration of measure for Gaussian measures. Using the method in [8] one can show the following (see [22, §18]):

Theorem 8.2. *If E and F are completely C_E and C_F-tight, respectively, then for any j.c.b. map $\varphi : E\times F\to\mathbb{C}$ and any $m\in\mathbb{N}$, $x=(x_j)\in E^m$, $y=(y_j)\in F^m$, and $t=(t_j)$ with $t_j>0$ one has*

$$|\sum_j \varphi(x_j,y_j)| \le C(\|x\|_R\|y\|_C + \|t\cdot x\|_C\|t^{-1}\cdot y\|_R),$$

with $C\le C_E C_F\|\varphi\|_{jcb}$.

Proposition 8.3. *If E is an exact operator space, then E is C_E-tight with $C_E\le\sqrt{2}ex(E)$.*

We need the following theorem:

Theorem 8.4 (Haagerup–Thorbjørnsen [9] + measure concentration)**.** *For any $\varepsilon>0$ there exists γ_ε such that for any $N,K,n\in\mathbb{N}$ and any $x_1,\ldots,x_m\in M_K$ one has*

$$\mathbb{E}\|\sum_{j=1}^n Y_j^{(N)}\otimes x_j\| \le (1+\varepsilon)\left[1+\gamma_\varepsilon\left(\frac{\log K}{N}\right)^{1/2}\right](\|x\|_R+\|x\|_C).$$

Proof of Proposition 8.3. Set $\beta(N) := (1+\varepsilon)[1+\gamma_\varepsilon(\frac{\operatorname{Log} K}{N})^{1/2}]$. Let $x_1,\ldots,x_n\in E$ be given and set $E_0 := \operatorname{span}\{x_1,\ldots,x_n\}$. Since E is C-exact, there exist $K\in\mathbb{N}$, $F\subset M_K$, and a c.b. map $u: E_0\to F$ such that $\|u\|_{cb}\|u^{-1}\|_{cb}\le C$. One has

$$\begin{aligned}\mathbb{E}\|\sum Y_j^{(N)}\otimes x_j\| &\le \|u^{-1}\|_{cb}\mathbb{E}\|\sum Y_j^{(N)}\otimes u(x_j)\|\\ &\le \|u^{-1}\|_{cb}\beta(N)(\|u(x)\|_R+\|u(x)\|_C)\\ &\le \|u^{-1}\|_{cb}\|u\|_{cb}\beta(N)(\|x\|_R+\|x\|_C)\\ &\le \sqrt{2}C\beta(N)(\|x\|_R^2+\|x\|_C^2).\end{aligned}$$

Since $\beta(N)$ tends to $1+\varepsilon$ as $N\to\infty$, we are done. □

Corollary 8.5. *If an operator space E and its dual space E^* are exact, then $E\cong R_I\oplus C_J\subset B(\ell_2(I)\oplus\ell_2(J))$ completely isomorphically.*

Proof. The identity map $E \to E$ is trivially completely bounded with c.b. norm 1. Thus, the canonical pairing $E \times E^* \to \mathbb{C}$ has the j.c.b. norm 1. By Theorem 8.2 and Proposition 8.3 $\mathrm{id}_E : E \to E$ factors though $R \oplus C$. By Oikhberg's result [16], E is completely isomorphic to some $R_I \oplus C_J$. $\square$

Example 8.6. The C*-algebra $\mathcal{C}$ is not exact. To see this, let $E_n := \mathrm{span}\{U_1, \dots, U_n]\} \subset \mathcal{C}$. We observe that $\|(U_j)\|_C = \|(U_j)\|_R = \sqrt{n}$. Thus, by the above theorem, one has $\lim_{N\to\infty} \mathbb{E}\| \sum_{j=1}^n Y_j^{(N)} \otimes U_j\| \le 2\sqrt{2n}\, ex(E_n)$. Note that $\sup_{\|a_j\|\le 1} \| \sum_j y_j \otimes a_j\| \le \| \sum_j y_j \otimes U_j\|$ for all $y_j \in M_N$. Thus, we have

$$\sum_j \tau_N(|Y_j^{(N)}|^2) \le \| \sum_j Y_j^{(N)} \otimes \overline{Y_j^{(N)}}\| \le \sup_i \|Y_i^{(N)}\| \| \sum_j Y_j^{(N)} \otimes U_j\|.$$

By the fact that $\lim_{N\to\infty} \sup_j \|Y_j^{(N)}\| = 2$, and

$$\mathbb{E}[\sum_{j=1}^n \tau_N(|Y_j^{(N)}|^2)] = N^{-1} \sum_{j=1}^n \sum_{k,l=1}^N \mathbb{E}[|Y^{(N)}(k,l)|^2] = n,$$

we have $n/2 \le 2\sqrt{2n}\, ex(E_n)$, and hence $ex(E_n) \ge \frac{\sqrt{n}}{4\sqrt{2}}$. This implies that $\mathcal{C}$ is not exact.

Definition 8.7. Let E and F be operator spaces with $\dim E = \dim F < \infty$ and $u : E \to F$ be a linear map. For each $N \in \mathbb{N}$, let $u_N : M_N(E) \to M_N(F)$ be as in Section 6. We set $d_N(E, F) := \inf\{\|u_N\|\|(u_N)^{-1}\| \mid u : E \to F : \text{isomorphism}\}$.

For any $C \ge 1$ and any operator space E with $\dim E < \infty$, we set

$$K_E(N, C) := \inf\{K \in \mathbb{N} \mid \exists F \subset M_K;\ d_N(E, F) \le C\}.$$

Definition 8.8. Let E be a finite dimensional operator space. We say that E is C-*subexponential* if one has $\limsup_{N\to\infty} N^{-1} \log K_E(N, C) = 0$. For a given operator space X, we say that X is C-*subexponential* if every finite dimensional subspace of X is C-subexponential. We set $subexp(X) := \inf\{C \mid X \text{ is } C\text{-subexponential}\}$.

Remark 8.9. By Theorem 8.4, one can show that any C-subexponential operator space X is $2C$-tight. Since $subexp(X) = subexp(M_n(X))$ for $n \ge 1$, X is completely $2C$-tight. Thus, the assertion of Theorem 8.2 holds for subexponential operator spaces. We also note that a finite dimensional operator space E is C-exact if and only if $\sup_N K_E(N, C) < \infty$ holds. Finally, we note that one has $d_{cb}(E, F) = \sup_N d_N(E, F)$ for $\dim E = \dim F < \infty$.

8.2. **Non-exact subexponential C*-algebras.** We will show that the class of subexponential operator spaces is strictly larger than the class of exact ones. Recall that $\{Y_j^{(N)}\}_{j\ge 1}$ is an i.i.d. sequence of $N \times N$-Gaussian random matrices on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For the sake of convenience we set $Y_0^{(N)} := 1$. For each $\omega \in \Omega$, let $Y_j(\omega) = (Y_j^{(N)}(\omega))_{N\ge 1} \in B := \prod_{N\ge 1} M_N$. Let $A(\omega)$ be the "random" C*-subalgebra of B generated by $\{Y_j(\omega) \mid j \ge 0\}$.

Theorem 8.10. *The* C*-*algebra* $A(\omega)$ *is* 1-*subexponential almost surely, but not exact.*

Remark 8.11. de la Salle showed the same assertion for uniformly distributed unitary random matrices.

Deterministic picture. We first consider the "non-random" case. We fix $n \in \mathbb{N}$ and $y_j = (y_j^{(N)})_{N\ge 1} \in B$ for $1 \le j \le n$. Set $y_0 := 1$ and $A := \mathrm{C}^*(y_j \mid 0 \le j \le n) \subset B$. Let C be a unital C*-algebra with faithful trace τ. Suppose that C is generated by $c_0, \dots, c_n$ with $c_0 = 1$. We denote by $\mathcal{P}$ the set of non-commutative $*$-polynomials in n variables, and by

$\mathcal{P}_d$ the polynomials in $\mathcal{P}$ of degree $\geq d$. We set $y := (y_j)_{j=1}^n \in B^n$, $y^{(N)} := (y_j^{(N)})_{j=1}^n \in M_N^n$, and $c := (c_j)_{j=1}^n \in C^n$.

For any n-tuple $z = (z_j)_{j=1}^n$ of elements in a C*-algebra A_0 and $P \in \mathcal{P}$ (resp. $m \geq 1$ and $P \in M_m \otimes \mathcal{P}$), we denote by $P(z)$ the elements in A_0 (resp. $M_m \otimes A_0$) given by substituting each z_j for the j-th variable of P.

We assume that $y^{(N)}$ converges to c *strongly* as $N \to \infty$, that is, for any $P \in \mathcal{P}$, we have

$$\lim_{N\to\infty} \tau_N(P(y^{(N)})) = \tau(P(c)), \quad \lim_{N\to\infty} \|P(y^{(N)})\| = \|P(c)\|.$$

For $d, m, t \in \mathbb{N}$, we define

$$C_d(t, m) := \sup_{N \geq m} \sup_{k \leq t} \{\|P(y^{(N)})\| \mid P \in M_k \otimes \mathcal{P}_d, \|P(c)\| \leq 1\}.$$

Theorem 8.12. *Suppose that for any* $d \geq 1$ *there exist constants* $a \geq 1$ *and* $D \geq 1$ *such that* $\lim_{N\to\infty} C_d(N, aN^D) = 1$*, and the* C*-*algebra* C *is exact. Then,* A *is* $(1+\varepsilon)$*-subexponential for all* $\varepsilon > 0$*.*

Proof. Let $E \subset A$ be a given finite dimensional subspace and $\varepsilon > 0$ be arbitrary. We will show that $\lim_{N\to\infty} N^{-1} \log K_N(E, 1+\varepsilon) = 0$. We may assume that $E \subset \mathcal{P}_d(y) := \{P(y) \mid P \in \mathcal{P}_d\}$ for some $d \in \mathbb{N}$. Then, we find $a > 0$ and $D > 0$ in such a way that $C_d(N, aN^D) \leq 1 + \varepsilon$ for all sufficiently large $N \geq 1$.

Let $\widehat{E}$ be the image of E by the canonical surjection $B \to \prod_{N' \geq aN^D} M_{N'}$. We denote by $\widehat{y}_j$ the element in $\widehat{E}$ corresponding to $y_j \in A$, and set $\widehat{y} := (\widehat{y}_j)_{j=1}^n$. Consider the map

$$u : \widehat{E} \ni P(\widehat{y}) \mapsto P(c) \in C \quad \text{for } P \in \mathcal{P}_d.$$

Note that $\|u\|_{cb} \leq 1$ and $F := u(\widehat{E})$ is a finite dimensional subspace of C and is independent of N. The condition $C_d(N, aN^D) \leq 1+\varepsilon$ implies that $\|(u_N : M_N(\widehat{E}) \to M_N(F))^{-1}\| \leq 1+\varepsilon$. Since C is 1-exact, there exist $K \in \mathbb{N}$ and $\widehat{F} \subset M_K$ such that $d_{cb}(F, \widehat{F}) \leq 1 + \varepsilon$. Therefore, there exists $G \subset (\prod_{N' < aN^D} M_{N'}) \oplus M_K$ such that $d_N(E, G) \leq (1+\varepsilon)^2$, which implies that $K_E(N, (1+\varepsilon)^2) \leq 2^{-1} a^2 N^{2D} + K$. □

Remark 8.13. In Theorem 8.12, if we further assume that $\sum_{j=1}^n \tau(|c_j|^2) > \|\sum_{j=1}^n y_j \otimes \bar{c}_j\|_{\min}$, then A is not exact. To see this, take an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let $M_{\mathcal{U}}$ be the ultraproduct of $(M_n, \tau_n)_{n\geq 1}$. Let $Q_1 : B \to M_{\mathcal{U}}$ be the quotient map and $V : A \hookrightarrow B$ be the inclusion map. By the assumption of strong convergence, we can show that there is a trace preserving embedding $C \subset M_{\mathcal{U}}$ satisfying that $Q_2 Q_1 V(y_j) = c_j$ for $1 \leq j \leq n$, where $Q_2 : M_{\mathcal{U}} \to C'' \subset M_{\mathcal{U}}$ is the trace preserving conditional expectation. Now suppose that A is exact. Then, $V \otimes \mathrm{id} : A \otimes_{\min} \bar{C} \to B \otimes_{\max} \bar{C}$ must be bounded. This implies that $\|\sum_j c_j \otimes \bar{c}_j\|_{\max} \leq \|\sum_{j=1}^n y_j \otimes \bar{c}_j\|_{\min}$, and hence one has $\sum_{j=1}^n \tau(|c_j|^2) \leq \|\sum_{j=1}^n y_j \otimes \bar{c}_j\|_{\min}$, which contradicts our assumption.

We now go back to the random C*-algebras $A(\omega)$, $\omega \in \Omega$. Firstly, we observe that $\sum_{j=1}^n \tau(|c_j|^2) = n$ and $\|\sum_{j=1}^n y_j \otimes \bar{c}_j\|_{\min} \leq \|(y_j)_j\|_R + \|(y_j)\|_C \leq 2\sqrt{n} \sup \|y_j\|$ hold when $(c_j)_j$ is a free circular system. In the unitary case y_j has norm 1, but in the Gaussian case we do *not* have $\sup_j \|Y_j(\omega)\|_B < \infty$ almost surely. However, each $Y_j(\omega)$ is a.s. in B and we still can estimate $\|\sum_{j=1}^n Y_j(\omega) \otimes \bar{c}_j\|$. Indeed, I show in my paper [24] that this is a.s. $O(\sqrt{n})$ when $n \to \infty$. So we conclude in this way that $A(\omega)$ is a.s. not exact.

Remark 8.14. Let E be an operator space E with $\dim E = n$ and let $C > 1$ and $N \in \mathbb{N}$. In [23], the following variant was introduced:

$$k_E(N, C) := \inf\{k \geq 1 \mid \exists F \subset \bigoplus_{i=1}^k M_N;\ d_N(E, F) \leq C\}.$$

Then, we can show that $K_E(N,C) \le (\frac{3C}{C-1})^{2nN^2}$ and $K_E(N,C) \le N k_E(N,C)$. Thus, one has $\log K_E(N,C) \le nN^2 C'$, where C' depends only on C. We do not know whether there exist operator spaces E such that $\log K_E(N,C)$ is intermediate between $O(N)$ and $O(N^2)$.

Using quantum expanders, we can show that there exists E such that $\log k_E(N,C) \approx N^2$. For example, this holds for $E = OH_n$ or $E = \mathrm{span}\{U_1,\dots,U_n\} \subset \mathcal{C}$. In fact, OH and $\mathcal{C}$ are not exact and also not subexponential.

References

[1] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[2] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor. The Grothendieck constant is strictly smaller than Krivine's bound. *Forum Math. Pi*, 1:e4, 42, 2013.

[3] A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \ne 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976.

[4] A. Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques. *Bol. Soc. Mat. São Paulo*, 8:1–79, 1953.

[5] U. Haagerup. The Grothendieck inequality for bilinear forms on C^*-algebras. *Adv. in Math.*, 56(2):93–116, 1985.

[6] U. Haagerup. A new upper bound for the complex Grothendieck constant. *Israel J. Math.*, 60(2):199–224, 1987.

[7] U. Haagerup and T. Itoh. Grothendieck type norms for bilinear forms on C^*-algebras. *J. Operator Theory*, 34(2):263–283, 1995.

[8] U. Haagerup and M. Musat. The Effros-Ruan conjecture for bilinear forms on C^*-algebras. *Invent. Math.*, 174(1):139–163, 2008.

[9] U. Haagerup and S. Thorbjørnsen. Random matrices and K-theory for exact C^*-algebras. *Doc. Math.*, 4:341–450 (electronic), 1999.

[10] M. Junge and G. Pisier. Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$. *Geom. Funct. Anal.*, 5(2):329–363, 1995.

[11] S. Khot and A. Naor. Grothendieck-type inequalities in combinatorial optimization. *Comm. Pure Appl. Math.*, 65(7):992–1035, 2012.

[12] E. Kirchberg. On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. *Invent. Math.*, 112(3):449–489, 1993.

[13] E. Kirchberg. Commutants of unitaries in UHF algebras and functorial properties of exactness. *J. Reine Angew. Math.*, 452:39–77, 1994.

[14] J.-L. Krivine. Sur la constante de grothendieck. *C. R. Acad. Sci. Paris Sér. A-B*, 284(8):A445–A446, 1977.

[15] J. Lindenstrauss and A. Pełczyński. Absolutely summing operators in L_p-spaces and their applications. *Studia Math.*, 29:275–326, 1968.

[16] Timur Oikhberg. Direct sums of operator spaces. *J. London Math. Soc. (2)*, 64(1):144–160, 2001.

[17] N. Ozawa. About the QWEP conjecture. *Internat. J. Math.*, 15(5):501–530, 2004.

[18] N. Ozawa. About the Connes embedding conjecture: algebraic approaches. *Jpn. J. Math.*, 8(1):147–183, 2013.

[19] A. Pietsch. *Operator ideals*, volume 20 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1980.

[20] G. Pisier. Grothendieck's theorem for noncommutative C^*-algebras, with an appendix on Grothendieck's constants. *J. Funct. Anal.*, 29(3):397–415, 1978.

[21] G. Pisier. *Factorization of linear operators and geometry of Banach spaces*, volume 60 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[22] G. Pisier. Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc. (N.S.)*, 49(2):237–323, 2012.

[23] G. Pisier. Quantum expanders and geometry of operator spaces. *J. Eur. Math. Soc. (JEMS)*, 16(6):1183–1219, 2014.

[24] G. Pisier. Random matrices and subexponential operator spaces. *Israel J. Math.*, 203(1):223–273, 2014.

[25] G. Pisier and D. Shlyakhtenko. Grothendieck's theorem for operator spaces. *Invent. Math.*, 150(1):185–217, 2002.

[26] O. Regev and T. Vidick. Elementary proofs of Grothendieck theorems for completely bounded norms. *J. Operator Theory*, 71(2):491–505, 2014.

[27] R. Schneider. Zonoids whose polars are zonoids. *Proc. Amer. Math. Soc.*, 50:365–368, 1975.

[28] B. S. Tsirelson. Quantum generalizations of Bell's inequality. *Lett. Math. Phys.*, 4(2):93–100, 1980.
[29] Q. Xu. Operator-space Grothendieck inequalities for noncommutative L_p-spaces. *Duke Math. J.*, 131(3):525–574, 2006.

Texas A&M University, College Station, TX 77843, U.S.A. and Université Paris VI IMJ, Equipe d'Analyse Fonctionnelle, Case 186, 75252 Paris Cedex 05, France

Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan